We know that generally:

$exp(z) = exp(zLoge) = exp(z(\log e + 2k\pi i)),\ k = 0, \pm 1, \pm 2, etc$
$exp(z) = exp(z(1 + 2k\pi i)),\ k = 0, \pm 1, \pm 2, etc \qquad (1)$

So the function represented by $exp(z)$ is infinitely multi-valued.
The value for $k = 0$ is called the principal value and is equal to e^z as defined by the series

$exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + ...$

When $z = 1$ in (1) the principal value (PV) of $exp(z)$ is obtained by setting $k = 0$ (which is equivalent to taking the PV of $Log\ e$) in (1) and gives:

$e^1 = e = 2.718...$

Consider the case where $z = 1 + 2\pi i$ and we take the principal value again then $exp(z) = e^{1+2\pi i} = e$.

As Penrose points out in section 5.4 of RTR:
if we choose a particular value for w^z, then we can multiply or divide this particular choice by $e^{z \cdot 2\pi i}$ and derive another allowable set of values for w^z.

So $e^{1+2\pi i} \cdot e^{(1+2\pi i) \cdot 2\pi i} = e^{(1+2\pi i)(1+2\pi i)}$ is another value. (2)

If we do this k times then the infinite set of values given by

$e^{1+2\pi i} \cdot e^{(1+2\pi i) \cdot 2k\pi i} = e^{(1+2\pi i)(1+2k\pi i)},\ k = 0, \pm 1, \pm 2, etc \qquad (3)$

are also allowable values.

The expression 3 can be re-written as:

$e^{(1+2\pi i)(loge+2k\pi i)},\ k = 0, \pm 1, \pm 2, etc$

and it can immediately be seen that (3) therefore represents all the values of $e^{1+2\pi i}$ and putting $k = 1$ gives (2), which is the value of $(e^{1+2\pi i})^{1+2\pi i}$ in our 'paradox'. Putting $k = 1$ amounts to using $loge = 1 + 2\pi i$ which is not the PV.

In fact (3) can be written as:

$e^{1+2\pi i+2k\pi i-4k\pi^2} = e^{1-4k\pi^2} \cdot e^{2\pi i(k+1)}$

and putting k=1 gives $e^{1+4\pi i-4\pi^2} = e^{1-4\pi^2}$ which is the result obtained in the exercise.

Notice also that if we put $k = 0$ in (3) then we obtain e which is what we would have expected to get in the exercise. So we can see that the 'fallacy' happens because we do not take principal values consistently.

We cannot say $e^{1+2\pi i} = (e^{1+2\pi i})^{1+2\pi i}$, because, as we have shown above, the LHS uses $log\ e = 1$ and the RHS uses $log\ e = 1 + 2\pi i$. The LHS and RHS represent 2 different values for the infinitely-valued $e^{1+2\pi i}$ and therefore cannot be equated.

Similarly, since $e^{1+2\pi i}$ is multi-valued, we cannot say $e = e^{1+2\pi i}$ unless we make it clear that we are using the PV of $e^{1+2\pi i}$, which is e.

If we always use the same value of $log\ e$ then our results will be consistent. As Penrose states, it is usual to use $log\ e = 1$, the PV, and then e^z is unambiguously defined for all z.