Road to Reality Exercise 5.15

I will use $Logz$ to denote the general logarithm of z and $logz$ to denote the principal value of $Logz$, so that

$Logz = logz + 2k\pi i, k = 0, \pm 1, \pm 2\ etc$

By definition $z^w = exp\{wLogz\}$ (1)

Consider $(z^a)^b$ and z^{ab} where z, a and b are complex.

Using (1) gives
$(z^a)^b = exp\{bLogz^a\}$ (2)
and
$z^{ab} = exp\{abLogz\}$ (3)

Simplifying the exponent $bLogz^a$ on the RHS of (2) gives
$b\{log[exp(aLogz)] + 2\pi in\}$
$= b\{aLogz + 2\pi in\}$
$= abLogz + 2\pi inb,\ n = 0 \pm 1, \pm 2,\ etc$ (4)

Putting (4) into (2) gives
$(z^a)^b = exp\{abLogz + 2\pi inb\},\ n = 0 \pm 1, \pm 2,\ etc$ (5)

From examining (3) and (5) it is clear that both are multi-valued and that although it is true that every value of (3) is equal to some value of (5), it is not true that every value of (5) is equal to some value of (3). So if we choose a value for Logz in (3), let's call it w, then for $(z^a)^b$ to be equal to z^{ab} we must have

$exp(abLogz + 2\pi inb) = exp(abw),\ n = 0 \pm 1, \pm 2,\ etc$
$\Rightarrow abLogz + 2\pi inb = abw + 2\pi ik,\ k, n = 0 \pm 1, \pm 2,\ etc$ $***$
$\Rightarrow b(Logz^a + 2\pi in) = abw + 2\pi ik,\ k, n = 0 \pm 1, \pm 2,\ etc$
$\Rightarrow bLogz^a = abw + 2\pi ik,\ k = 0 \pm 1, \pm 2,\ etc$
$\Rightarrow Log(z^a)^b - 2\pi ik = abw,\ k = 0 \pm 1, \pm 2,\ etc$
$\Rightarrow Log(z^a)^b + 2\pi ik = abw,\ k = 0 \pm 1, \pm 2,\ etc$
$\Rightarrow Log(z^a)^b = abw \Rightarrow bLogz^a = abw \Rightarrow Logz^a = aw$
where w is equal to the value of $Logz$ chosen for the RHS of $(z^a)^b = z^{ab}$

Note: This last section could have been shown directly by equating (2) and (3) after substituting w for $Logz$ in (3), but it serves as a check that (5) is correct.

Interesting aside:
Since from (5) $(z^b)^a = abLogz + 2\pi ima,\ m = 0 \pm 1, \pm 2,\ etc$
it is clear that in general $(z^a)^b \neq (z^b)^a$.
However in the particular case where principal values of the logarithms are used then $(z^a)^b = (z^b)^a = z^{ab}$, since all three expressions are equal to $exp(ablogz)$, which is single-valued.

*** Explanation for this step:
$exp(z_1) = exp(z_2)$
$\Rightarrow exp(z_1 - z_2) = 1$
$\Rightarrow z_1 - z_2 = 2\pi ik,\ k = 0, \pm 1, \pm 2,\ etc$