Solution to Exercise [8.7]

by Tony Dean

I assume that exercise 8.7 consists in showing that the bilinear form

$$f(z)=\frac{z-1}{iz+i},$$

as given near the end of section 8.2, effects a 90° rotation of the real circle in the Riemann sphere onto the unit circle. However, note that $f(0)=i$, $f(1)=0$ and $f(-1)=\infty$. Thus, while f appears to rotate the North pole of the real circle onto i, as we might expect, yet it does not leave 1 and -1 fixed, as we would also expect. Being suspicious at this point, I argued that if f were indeed a 90° rotation, applying it four times would leave the sphere fixed; i.e., we should have $f^4(z)=f\left(f\left(f\left(f(z)\right)\right)\right)=z$. However, note that $f^3(z)=z$ instead:

$$f^2(z)=\frac{\left(\frac{z-1}{iz+i}\right)-1}{i\left(\frac{z-1}{iz+i}\right)+i}=\frac{z-1-iz-i}{iz-i-z-1}=-\frac{(1-i)z-(1+i)}{(1-i)z+(1+i)}$$

If we multiply both numerator and denominator by $1+i$, we get

$$f^2(z)=-\frac{2z-2i}{2z+2i}=-\frac{z-i}{z+1},$$

because $(1-i)(1+i)=2$ and $(1+i)^2=2i$. Then,

$$f^3(z)=f^2\left(f(z)\right)=-\frac{\left(\frac{z-1}{iz+i}\right)-i}{\left(\frac{z-1}{iz+i}\right)+i}=-\frac{z-1+z+1}{z-1-z-1}=-\frac{2z}{-2}=z$$

This shows that f cannot be a 90° rotation.

So, what is the correct formula? We're looking for a bilinear function

$$g(z)=\frac{az+b}{cz+d}$$

which effects a clockwise 90° rotation. So at a minimum, it must satisfy the conditions

$$
\begin{aligned}
g(0) &= i \\
g(-i) &= 0 \\
g(1) &= 1
\end{aligned}
\qquad (1)
$$

These conditions imply

$$
\begin{aligned}
&i = b/d \Rightarrow b = di \\
&0 = \frac{-ai+b}{-ci+d} \Rightarrow b = ai \Rightarrow a = d \\
&1 = \frac{d+di}{c+d} \Rightarrow c = di
\end{aligned}
$$

Therefore:

$$
g(z) = \frac{dz+di}{diz+d} = \frac{z+i}{iz+1}.
$$

Is this a rotation? Well, we know that there is a rotation satisfying (1) and that is has to be a bilinear form. The only bilinear form that satisfies the conditions (1) is g, so it must be a rotation.