The equation of a circle in the z-plane with centre at z_0 and radius vector r, where $|r|$ =constant, is

$$z = z_0 + r \qquad (1)$$

So $z - z_0 = r$, and $(z - z_0)(\overline{z - z_0}) = (z - z_0)(\overline{z} - \overline{z}_0) = r\overline{r} = |r|^2 \qquad (2)$

So now let's apply the transformation $w = 1/z$ to equation (2):
$(1/w - z_0)(1/\overline{w} - \overline{z}_0) = |r|^2$
$\Rightarrow (1 - wz_0)(1 - \overline{wz}_0) = |r|^2 w\overline{w}$
$\Rightarrow 1 - wz_0 - \overline{wz}_0 + w\overline{w}z_0\overline{z}_0 = |r|^2 w\overline{w}$

Rearranging gives:

$$(|r|^2 - z_0\overline{z}_0)w\overline{w} + \overline{wz}_0 + wz_0 - 1 = 0 \qquad (3)$$
$$\Rightarrow w\overline{w} + \frac{\overline{z}_0}{|r|^2 - z_0\overline{z}_0}\overline{w} + \frac{z_0}{|r|^2 - z_0\overline{z}_0}w = \frac{1}{|r|^2 - z_0\overline{z}_0} \qquad (4)$$

as long as $|r|^2 - |z_0|^2 \neq 0$. Geometrically this means that the circle does not pass through through the origin of the z-plane. See below for the case when $|r|^2 - |z_0|^2 = 0$.

If we define $\alpha = |r|^2 - z_0\overline{z}_0 = |r|^2 - |z_0|^2$ and also $w_0 = \dfrac{-\overline{z}_0}{|r|^2 - |z_0|^2} = -\dfrac{\overline{z}_0}{\alpha}$ then $w_0\overline{w}_0 = \dfrac{z_0\overline{z}_0}{\alpha^2}$ and (4) becomes

$$(w - w_0)(\overline{w} - \overline{w}_0) - w_0\overline{w}_0 = -\frac{w_0}{\overline{z}_0}$$
$$(w - w_0)(\overline{w} - \overline{w}_0) = w_0\overline{w}_0 - \frac{w_0}{\overline{z}_0}$$
$$(w - w_0)(\overline{w} - \overline{w}_0) = \frac{|r|^2}{\alpha^2} = s^2 \qquad (5)$$

Equation (5) is of the same form as equation (2), and represents the transformed circle in the w-plane with centre at

$$w_0 = \frac{-\overline{z}_0}{\alpha} \text{ and radius } s = \sqrt{\frac{|r|^2}{\alpha^2}} = \frac{|r|}{|\alpha|}$$

When $|r|^2 - |z_0|^2 = 0$, equation (3) reduces to:

$\overline{wz}_0 + wz_0 - 1 = 0$

Substituting $w = u + iv$, and $z_0 = x_0 + iy_0$, gives:

$2ux_0 - 2vy_0 = 1$

This is the equation of a straight line in the w-plane. For circles in the z-plane that pass through the origin with centre on the x axis ($y_0 = 0$), the equation reduces to $2ux_0 = 1$, which are straight lines parallel to the v-axis. For circles in the z-plane that pass through the origin with centre on the y axis ($x_0 = 0$), the equation reduces to $2vy_0 = -1$, which are straight lines parallel to the u-axis.

All other circles in the z-plane which pass through the origin are transformed to straight lines according to the equation:

$v = u(x_0/y_0) - 1/y_0$.