Another possible proof:

$$\text{Log}(a^b) = b\log{(a)} + ibk(2\pi) + in(2\pi) \tag{1}$$

(where Log is multivalued and log is principle, converse to Wikipedia, but c.f. Wikipedia complex powers or I can derive this if someone is interested)

Now consider the validity of $(e^{(1+i2\pi)})^{(1+i2\pi)} = e^{[(1+i2\pi)(1+i2\pi)]}$. First I will look at the LHS taking logarithms:

$$\begin{aligned}
&\text{Log}\left(e^{(1+i2\pi)})^{(1+i2\pi)}\right) \\
&= (1+i2\pi)\log{(e^{(1+i2\pi)})} + i(1+i2\pi)\alpha(2\pi) + i\beta(2\pi) \\
&= (1+i2\pi)(1+i2\pi) + i\alpha(2\pi + i4\pi^2) + i\beta(2\pi) \\
&= 1 + i4\pi - 4\pi^2 + i(2\pi)\alpha - \alpha 4\pi^2 + i(2\pi)\beta \\
&= [1 - 4\pi^2 - \alpha 4\pi^2] + i(2\pi)[2 + \alpha + \beta]
\end{aligned} \tag{2}$$

were in the above α, β are integers determining the branch we are choosing.

Next consider the RHS:

$$\begin{aligned}
&\text{Log}\left(e^{(1+i2\pi)(1+i2\pi)}\right) \\
&= (1+i2\pi)^2\log{(e)} + i(1+i2\pi)^2\sigma(2\pi) + i\rho(2\pi) \\
&= 1 + i4\pi - 4\pi^2 + i(2\pi)\sigma - 8\pi^2\sigma - 8\pi^3 i\sigma + i\rho(2\pi) \\
&= [1 - 4\pi^2 - 8\pi^2\sigma] + i(2\pi)[2 + \sigma - 4\pi^2\sigma + \rho]
\end{aligned} \tag{3}$$

were again in equation (3) σ, ρ are integers (generally different than those in (2)).

But now if we demand agreement between the LHS and RHS is this possible? equating the real parts first:

$$\begin{aligned}
&1 - 4\pi^2 - \alpha 4\pi^2 = 1 - 4\pi^2 - 8\pi^2\sigma \\
&\alpha 4\pi^2 = 8\pi^2\sigma \\
&\alpha = 2\sigma
\end{aligned} \tag{4}$$

OK, then the imaginary parts:

$$\begin{aligned}
&2 + \alpha + \beta = 2 + \sigma - 4\pi^2\sigma + \rho \\
&2 + 2\sigma + \beta = 2 + \sigma - 4\pi^2\sigma + \rho \text{...(after using result (4))} \\
&\sigma + \beta = -4\pi^2\sigma + \rho \\
&\sigma(1 + 4\pi^2) = \rho - \beta \\
&\sigma = \frac{[\rho - \beta]}{(1 + 4\pi^2)}
\end{aligned} \tag{5}$$

But the point now is that $\alpha, \beta, \sigma, \rho$ are all supposed to be integers; it is clear from (5) that no matter what values you choose for ρ and β, σ will be non-integer, except! if you choose $\beta = \rho$ so $\sigma = 0$ and by equation (4) thus, $alpha = 0$.

OK so by demanding (1) we've been forced into

$$\mathrm{Log}\left(e^{(1+i2\pi)}\right)^{(1+i2\pi)}\Big) = [1 - 4\pi^2] + i(2\pi)[2+\beta]..(\text{subbing in our } \alpha = 0) \tag{6}$$

So far so good, but according the paradox we also want $e^{(1+2\pi i)} = \mathrm{Log}(e^{(1+i2\pi)})^{(1+i2\pi)}$ to hold. We have already expanded the RHS of this, so just the left to expand now:

$$\begin{aligned} &\mathrm{Log}\left(e^{(1+i2\pi)}\right) \\ &= (1+i2\pi)\log(e) + i(1+i2\pi)\delta(2\pi) + i\epsilon(2\pi) \\ &= (1+i2\pi) + i\delta(2\pi + i4\pi^2) + i\epsilon(2\pi) \\ &= 1 + i2\pi + i(2\pi)\delta - \delta 4\pi^2 + i(2\pi)\epsilon \\ &= [1 - 4\pi^2\delta] + i(2\pi)[1 + \delta + \epsilon] \end{aligned} \tag{7}$$

Seeking agreement between (6) and (7) means we must have $\delta = 1$, and from the imaginary parts also $1 + \delta + \epsilon = 2 + \beta$ which implies $2 + \epsilon = 2 + \beta$ or in other words $\epsilon = \beta$. So we can write:

$$\begin{aligned} &\mathrm{Log}\left(e^{(1+i2\pi)}\right) \\ &= [1 - 4\pi^2] + i(2\pi)[2+\beta] \end{aligned} \tag{8}$$

But now finally the paradox also demands that $e = e^{(1+i2\pi)}$. Taking logs of both sides using (8):

$$\begin{aligned} \mathrm{Log}\,(e) &= [1 - 4\pi^2] + i(2\pi)[2+\beta] \\ 1 + i\gamma(2\pi) &= [1 - 4\pi^2] + i(2\pi)[2+\beta] \end{aligned} \tag{9}$$

Comparing the real parts shows no consistency is possible, thus no matter which branches we choose throughout the paradox conistency between all of the equalities is not possible.