Exercise [3.4]

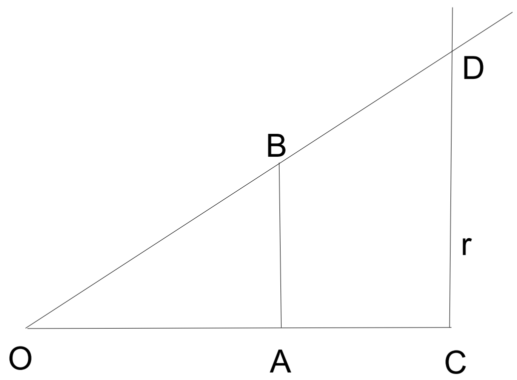
A preliminary problem

First, we need to solve a preliminary problem: given three segments having length *a,b,c* find a fourth segment having length *d* such that *a:b = c:d* .

A solution is the following: with reference to the figure, draw a right-angled triangle OAB, where OAB is the right angle, with side OA having length *a* and side AB having length *b*. Then extend [(*)] the line OA up to C, with OC having length *c.* Then send from C the parallel *r* to AB, that will intersect the line OB in a point D. The length of segment CD is *d*.

Note that intersection D always exists: otherwise line OB and *r* would be parallel, but being *r* parallel to AB also OB and AB would be parallel; but this is not true, because they intersect in B.

The triangles OAB and OCD are similar, having equal the angles BOA and DOC (common angles) and also the angles OAB and OCD (both right); therefore also the third angles are equal and the triangles are similar. Hence the corresponding sides are proportional: OA:AB= OC:CD, i.e. *a:b = c:d* .



A preliminary theorem: if $b:a=d:c$ *then* $a:b=c:d$

Let's assume that $a:b\neq c:d$; it would be either $a:b>c:d$ or $a:b<c:d$.

If $a:b>c:d$, it would exist (by the definition of > reported in the book) a pair of integers M and N such that

$$M\,a>N\,b \quad \text{and} \quad N\,d>M\,c$$

Renaming $M=N'$ and $N=M'$ these inequalities can be rewritten as

$$M'b<N'a \quad \text{and} \quad N'c<M'a$$

and this would mean that $(b:a)<(d:c)$; this is false, because the hypothesis is $b:a=d:c$.

In same way one can prove that $a:b<c:d$ would imply $(b:a)>(d:c)$, that is also false.

Hence it must be $a:b=c:d$.

Sum: $(a:b)+(c:d)=(x:y)$

Using the construction of the preliminary problem, find the segment with length *f* such that *d:c =b:f;* this means, because of theorem (1) , that *c:d=f:b* .

It is therefore $(a:b)+(c:d)=(a:b)+(f:b)=((a+f):b)$

Therefore we can define $x=a+f$, $y=b$, and this requires only the sum of segments.

Product: $(a:b)*(c:d)=(x:y)$

Using the construction of the preliminary problem, find a segment with length *f* such that *c:d=b:f*.

It is $(a:b)*(c:d)=(a:b)*(b:f)=(a:f)$ so we can set *x=a* and *y=f.*

(*) This assumes c>a; if c<=a, you can take the point C inside the segment OA, without needing to extend it.