Exercise 10.10
Part a
It's important to understand that complex conjugation is an operation that cannot be achieved by using the normal rules of arithmetic for complex numbers - it is a separate operation.

Given that Φ is holomorphic in $z = x + iy$ on the (x, y) patch then we can say that $\Phi = f(z)$, where $f(z)$ is a holomorphic function of z.
Given also that on the overlapping (X, Y) patch, z is holomorphic in $Z = X + iY$, we can say that $z = F(Z)$ where $F(Z)$ is a holomorphic function of Z.
By applying the mapping $z = F(Z)$ on the overlap we can say Φ is a function of Z, that is $\Phi = G(Z)$.
Since, as Penrose states on page 180 of RTR, holomorphic functions are those built up from the operations of addition and multiplication as applied to complex numbers, together with the procedure of taking a limit, we can see that this excludes the possibility of complex conjugation ($\bar{z}$ or $\bar{Z}$) appearing in the definitions of f and F.
This means that when $f(z)$ is transformed to $G(Z)$ by the mapping $z = F(Z)$, the function $G(Z)$ also cannot contain the complex conjugation operation and is therefore a holomorphic function. Therefore $\Phi = G(Z)$ is a holomorphic function of Z.

Part b
In chapter 8 section 2 on page 139 Penrose states that in a conformal map infinitesimal shapes are preserved. For example, infinitesimal circles drawn in the plane can be expanded or contracted by the mapping, but not distorted into ellipses. For the map to be holomorphic the circles must also not be reflected, i.e. turned over. So if a circle in one plane is traversed in an anticlockwise manner the image points of the circle in the transformed plane must also be traversed in an anticlockwise manner.
So referring to part a above, the holomorphic mapping $\Phi = f(z)$ to the Φ-plane, preserves the shape and orientation of infinitesimal circles in the z-plane. Similarly the holomorphic mapping $z = F(Z)$ to the z-plane, preserves the shape and orientation of infinitesimal circles in the Z-plane.
So infinitesimal circles in the Z-plane are preserved in the z-plane and these same circles in the z-plane are preserved in the Φ-plane by the holomorphic mapping $f(z)$.
Therefore the mapping from the Z-plane to the Φ-plane also preserves infinitesimal circles and is therefore holomorphic.

Part c
Given that
$\Phi = f(z) = f_1(x, y) + if_2(x, y)$,
$z = F(Z) = F_1(X, Y) + iF_2(X, Y) = x + iy$,
$\Phi = G(Z) = G_1(X, Y) + iG_2(X, Y)$,
where $\Phi = f(z)$ and $z = F(Z)$ are holomorphic in z and Z respectively, show that $\Phi = G(Z)$ is holomorphic in Z.

Since $\Phi = f(z)$ and $z = F(Z)$ are holomorphic in z and Z respectively, then using the Cauchy-Riemann equations on each function gives

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \; \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}$$
$$\frac{\partial F_1}{\partial X} = \frac{\partial F_2}{\partial Y}, \; \frac{\partial F_1}{\partial Y} = -\frac{\partial F_2}{\partial X} \text{ or } \frac{\partial x}{\partial X} = \frac{\partial y}{\partial Y}, \; \frac{\partial x}{\partial Y} = -\frac{\partial y}{\partial X}$$

Now $\Phi = f(z) = G(Z)$, so $f_1(x, y) = G_1(X, Y)$ and $f_2(x, y) = G_2(X, Y)$
So

$$\frac{\partial G_1}{\partial X} = \frac{\partial f_1}{\partial X} = \frac{\partial f_1}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial f_1}{\partial y}\frac{\partial y}{\partial X} = \frac{\partial f_2}{\partial y}\frac{\partial y}{\partial Y} + \frac{\partial f_2}{\partial x}\frac{\partial x}{\partial Y} = \frac{\partial f_2}{\partial Y} = \frac{\partial G_2}{\partial Y}$$
$$\frac{\partial G_1}{\partial Y} = \frac{\partial f_1}{\partial Y} = \frac{\partial f_1}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial f_1}{\partial y}\frac{\partial y}{\partial Y} = -\left[\frac{\partial f_2}{\partial y}\frac{\partial y}{\partial X} + \frac{\partial f_2}{\partial x}\frac{\partial x}{\partial X}\right] = -\frac{\partial f_2}{\partial X} = -\frac{\partial G_2}{\partial X}$$

Therefore

$$\frac{\partial G_1}{\partial X} = \frac{\partial G_2}{\partial Y} \text{ and } \frac{\partial G_1}{\partial Y} = -\frac{\partial G_2}{\partial X}$$

which are the Cauchy-Riemann equations for $\Phi = G(Z)$.
Therefore Φ is holomorphic in Z.