Begin Exercise 10.7

To find A and B in terms of a and b, we start by setting equal the general forms of the vector fields $\boldsymbol{\xi}$ in the (x, y) and the (X, Y) coordinate systems in a left-right order that suggests passing from the (x, y) to the (X, Y) coordinate systems. We are developing a way to extend the vector field $\boldsymbol{\xi}$ from an (x, y) coordinate patch to an overlapping (X, Y) coordinate patch.

$$\boldsymbol{\xi} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} = A\frac{\partial}{\partial X} + B\frac{\partial}{\partial Y}$$

Next, we need to develop expressions to substitute for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ that appear in the (x, y) version of $\boldsymbol{\xi}$ (the middle part of the three way equation). We can do this by taking the partial derivatives with respect to x and y of the *transition functions.*

$X = X(x, y)$

$Y = Y(x, y)$

The partial derivatives with respect to x and y are:

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y}$$

$$\frac{\partial}{\partial y} = \frac{\partial X}{\partial y}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y}$$

Substituting these partial derivatives into the expression for $\boldsymbol{\xi}$ we get.

$$\boldsymbol{\xi} = a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y} = a(\frac{\partial X}{\partial x}\frac{\partial}{\partial X}+\frac{\partial Y}{\partial x}\frac{\partial}{\partial Y})+b(\frac{\partial X}{\partial y}\frac{\partial}{\partial X}+\frac{\partial Y}{\partial y}\frac{\partial}{\partial Y}) = A\frac{\partial}{\partial X}+B\frac{\partial}{\partial Y}$$

Now,

$$\boldsymbol{\xi} = a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y} = a\frac{\partial X}{\partial x}\frac{\partial}{\partial X}+a\frac{\partial Y}{\partial x}\frac{\partial}{\partial Y}+b\frac{\partial X}{\partial y}\frac{\partial}{\partial X}+b\frac{\partial Y}{\partial y}\frac{\partial}{\partial Y} = A\frac{\partial}{\partial X}+B\frac{\partial}{\partial Y}$$

Rearranging to associate terms in $\frac{\partial}{\partial X}; \frac{\partial}{\partial Y}$:

$$\boldsymbol{\xi} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} = (a\frac{\partial X}{\partial x} + b\frac{\partial X}{\partial y})\frac{\partial}{\partial X} + (a\frac{\partial Y}{\partial x} + b\frac{\partial Y}{\partial y})\frac{\partial}{\partial Y} = A\frac{\partial}{\partial X} + B\frac{\partial}{\partial Y}$$

From which we obtain A and B in terms of a and b, as requested:

$$A = a\frac{\partial X}{\partial x} + b\frac{\partial X}{\partial y}$$

$$B = a\frac{\partial Y}{\partial x} + b\frac{\partial Y}{\partial y}$$

From these equations we can optain a and b in terms of A and B by analogy:

$$a = A\frac{\partial x}{\partial X} + B\frac{\partial x}{\partial Y}$$

$$b = A\frac{\partial y}{\partial X} + B\frac{\partial y}{\partial Y}$$

End of Exercise 10.7