Exercise [16.13]: General form of Cantor's "diagonal slash" argument

Given two sets A and B, where $|A| \leq |B|$ we wish to prove that $|A| < |B|$.[*] It suffices to show that there is no surjection[†] from A onto B. The "diagonal slash" argument, when applicable, demonstrates precisely this.

The key to the "diagonal slash" argument is to first find a bijection between B and some set $F \subseteq X^A$, for some (arbitrary) set X. Recall that X^A is the set of all functions from A to X. So each element b of B must correspond to some particular function $f_b: A \rightarrow X$. Let G be this bijection, so that $G(b) = f_b$ and $G^{-1}(f_b) = b$.

Then we must define a permutation function $p: X \rightarrow X$ such that $p(x) \neq x$ for all $x \in X$. Note that when $\mathrm{F} = \mathrm{X}^\mathrm{A}$, *any* p with this property will do. However, if $F \subset X^A$ then we may have to choose p more carefully; the criterion for choosing p will be explained below.

Now we can proceed with the actual argument.

Let $S : A \rightarrow B$ be any function from A to B. Construct the function $f_q \in F$ as follows:

$$f_q(a) = p(f_{S(a)}(a))$$

Where of course $f_{S(a)}$ is simply another way of writing $G(S(a))$. We require f_q thus constructed to be an element of F, and if $F \subset X^A$ we may need to choose p carefully to ensure that it is. This allows us to define $q = G^{-1}(f_q) \in B$.

Let $a \in A$. Then $f_q(a) = p(f_{S(a)}(a)) \neq f_{S(a)}(a)$, so $f_q \neq f_{S(a)}$. Applying G^{-1} to both sides yields $q \neq S(a)$. Since this is true for any $a \in A$, $q \notin S(A)$ and hence $S(A) \subset B$, i.e. S is not a surjection.

Therefore, no function from A to B is a surjection[‡].

■

[*] Note that I'm using the notation where $|\cdot|$ denotes the cardinality of a set.

[†] A function $f: X \rightarrow Y$ is *injective* if it is one-to-one; *surjective* if for every $y \in Y$ there's at least one $x \in X$ such that $y = f(x)$; and *bijective* if both are true (in which case it's a one-to-one map between sets X and Y).

[‡] The argument also works if we restrict S to *injective* functions only, thus showing that no *bijection* from A to B exists. This weaker version could *conceivably* be required to ensure $f_q \in F$ for some A, B, F, and p, if that constraint can't be met for the version shown… but otherwise there's no advantage to making this restriction.

If the above all sounds like gobbledygook, a couple of examples might clarify it!

First, take the original example in the book, demonstrating that $|A|<|\textit{set of subsets of } A|$, for any (arbitrary) set A. Then $B = \{\textit{all subsets of } A\}$, $X = \{0, 1\}$, and $F = X^A = \{ \textit{ all functions } f\text{: } A \rightarrow \{0, 1\} \}$. Now as discussed in the book, there's a bijection between B and F: For any $b \in B$, take $f_b = G(b)$ to be defined by $f_b(a) =$ (1 if $a \in b$ and 0 if not). i.e. $G(b)$ is just the *membership function* of the subset b (of A). There's only one possible choice for p, namely the function that switches 1 with 0, i.e. $p(1)=0$ and $p(0)=1$.

Having established all this, we can now apply the "diagonal slash" argument, by constructing f_q:

$$f_q(a) = p(f_{S(a)}(a)) = (0 \text{ if } a \in \mathrm{S}(a) \text{ and 1 if not})$$

And thus $q = G^{-1}(f_q) = \{a\text{: } a \notin S(a)\}$, which is the same subset of A as was described in the text (denoted there by "Q" instead). Showing that there's no $a \in A$ for which $S(a) = q$ completes the argument, both here and in the book.

For the second example, take $A = \mathbb{N}-0$ and $B = (0,1) \subset \mathbb{R}$. ((0,1) is the subset of real numbers strictly between 0 and 1). Any number in B can be represented as a sequence of decimal digits following a decimal point, and any *sequence* of digits can be considered as a function from n, the position in the sequence, to the digit at that (n^{th}) position. So take $X = \{0,...,9\}$. For any $b \in B$, take $f_b = G(b)$ to be the function in X^A that corresponds (as just described) to b's decimal expansion. However, there's a little difficulty here of a kind we didn't encounter in our last example. We require G to be a *bijection* from B to F, but certain real numbers have two possible decimal expansions: Any number who's decimal expansion ends in an infinite sequence of 0's can also be written using a decimal expansion that ends in an infinite sequence of 9's, and visa-versa. E.g.:

$$0.123050000000\ldots \equiv 0.123049999999\ldots$$

To overcome this problem, we simply choose to always use the first form, and disallow any decimal expansion that ends in an infinite sequence of 9's. We must also exclude the decimal sequence that contains all 0's from consideration, since that just equals 0, and 0 is not an element of B. Thus in this case $F \subset X^A$. We'll need to be sure that f_q is in fact in F when we construct it, and an easy way to ensure we don't "accidentally" produce one of these excluded sequences is to choose a p that never yields either 0 or 9. There are many possibilities, but for example $p(x) =$ (7 if $x<5$ and 2 otherwise) is as good a choice as any.

To proceed with the "diagonal slash argument" in this case, we take any infinite list of real numbers in (0,1), such that $S(a)$ yields the a^{th} number in the list. The function f_q then corresponds to the decimal sequence constructed by taking the first digit of the (decimal expansion of) the first number in the list, then the second digit of the second number in the list, and so on, and applying p to each of these digits in turn to generate a *new* decimal expansion (in this case composed solely of 2's and 7's) that differs by at least one digit from *every* number in the list – thus proving that no such list can contain every number in B! …And thus that (since $|(0,1)| \equiv \mathrm{C}$), the reals are not countable ($\mathrm{C} \neq \aleph_0$).