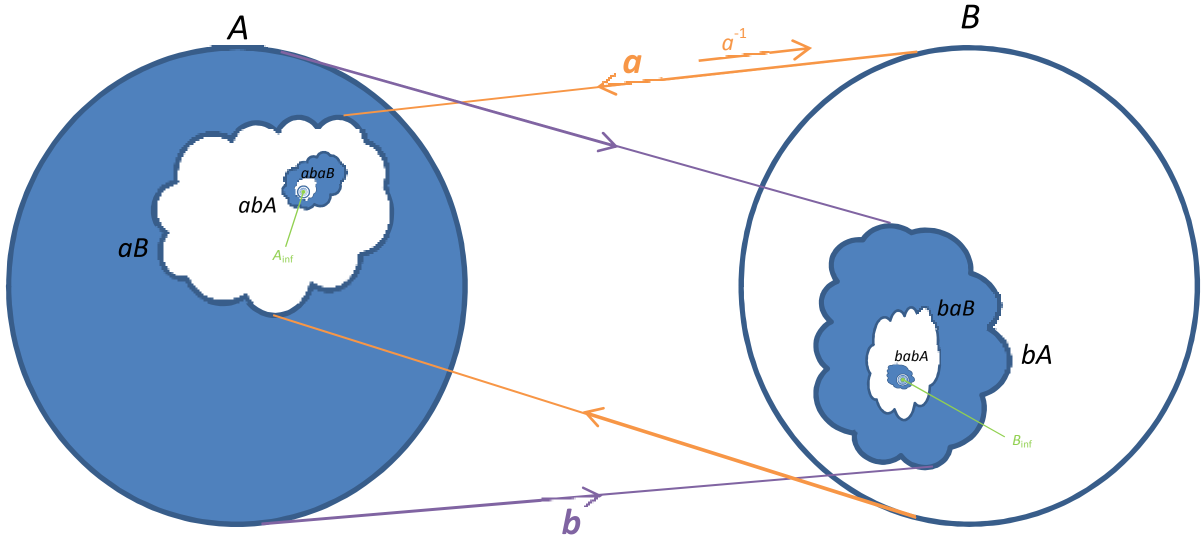
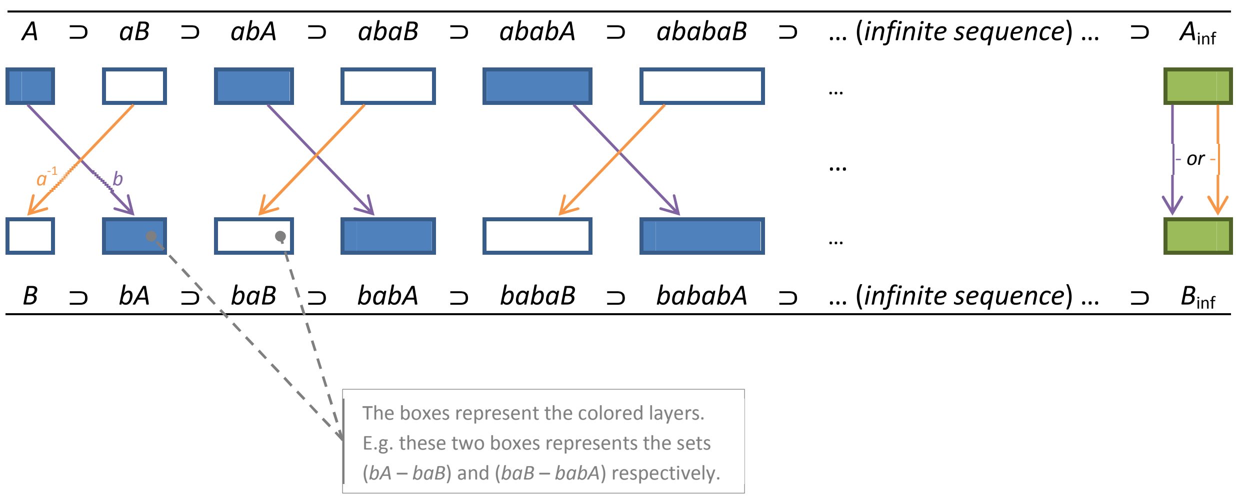
Exercise [16.10]

Venn Diagram:



The 1-1 correspondence (bijection) from *A* to *B*:



The Venn diagrams of both A and B can each be divided into alternating disjoint concentric layers (shown here in blue and white). Since mappings a and b are both 1-1, this means that layers of the same color all have the same cardinality. (They are all images of the sets A-aB and B-bA – for blue and white layers repectively – under repeated alternating application of a and b). Therefore, since a and b can continue to be applied indefinitely, there are an infinite number of such concentric layers.

The limit of the union of all such layers (within either set, A or B) may or may not encompass the entire set. If it does not encompass the entire set, that means that in the Venn diagram there is a "center" inside *all* of the (infinitely many) concentric layers. Call these "centers" A_{inf} and B_{inf}. They represent the (largest) subsets of A and B (respectively) that are *unchanged* by application of ab or ba (respectively). That is: $abA_{\text{inf}} = A_{\text{inf}}$ and $baB_{\text{inf}} = B_{\text{inf}}$. (NB these mappings need not be identity mappings when applied to the *individual elements* of A_{inf} or B_{inf}, just so long as they take these *sets* to themselves). Also, by construction, $A_{\text{inf}} = \text{a}B_{\text{inf}}$ and $B_{\text{inf}} = bA_{\text{inf}}$. Thus either a^{-1} or b will do equally well as a 1-1 map (bijection) from A_{inf} onto (all of) B_{inf}.

That gives us the required 1-1 map between A_{inf} and B_{inf}. We now need to extend this map to apply to the remainder of A and B – the parts of these sets within the concentric layers. We can do this by taking these layers in pairs from both A and B, and mapping the pairs of layers to each other "the wrong way around" – i.e. the inside layer from A to the outside layer from B, and visa-versa. This maps layers of the same color together. The diagram on the preceding page shows how this is done. In both sets there are an infinite number of layer pairs, so we start by mapping the outermost pair in A to the outermost pair in B, then map the next-from-outermost pair in A to the next-from-outermost pair in B, and so on. The result is a complete 1-1 map between the entire sets A and B.

■

Note that although we have proved the theoretical *existence* of such a map, actually *applying* it to a given element – say in A, to get the mapped element in B – requires us to determine the class that element belongs to: i.e. whether it is in a blue or a white layer, or within A_{inf}. There may or may not be an easy way of determining this (a shortcut): That will depend on the nature of the functions a and b. *If there are no shortcuts*, then the only remaining method is to repeatedly apply a^{-1} and b^{-1} to the element (in sequence) until it can no longer be done any more (because the result has fallen outside the domain of the inverse function we're trying to apply). In that case it's a "blue" element if the sequence ends inside A, and a "white" element if the sequence ends inside B: So then we'd apply either a^{-1} or b (respectively), to the original element, to get the correctly mapped element in B.

Unfortunately, however, we cannot (in general) know in advance how long this sequence will go for until it ends. If the point we are trying to map happens to be in A_{inf}, it will *never* end. Thus, using this method, there is no *finite* calculation that is guaranteed to find the correct mapping from an element in A to its image in B. Therefore:

If there is no shortcut to determining an element's type (blue, white, or "inf"), then the 1-1 map between A and B that we have constructed above is **not computable**[*].

[*] See pp374-376 of the book for a brief discussion of computability.