Exercise [16.06]

The first part of this exercise asks us to show that the associator A(a,b,c) = $a(bc)$-$(ab)c$ is antisymmetric for $a,b,c \in \{\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_6\}$. That is:

(1) A(a,b,c) + A(b,a,c) = 0 ⇔ $a(bc) + b(ac) - (ab + ba)c = 0$

(2) A(a,b,c) + A(a,c,b) = 0 ⇔ $a(bc + cb) - (ab)c - (ac)b = 0$

(3) A(a,b,c) + A(c,b,a) = 0 ⇔ $a(bc) - (cb)a + c(ba) - (ab)c = 0$

There are 7^3 = 343 possible ways of choosing values for a, b, and c. However, there are two symmetries we can use to reduce the number of combinations. They are:

(i) Cyclic symmetry: $\mathbf{i}_0 \to \mathbf{i}_1 \to \mathbf{i}_2 \to \mathbf{i}_3 \to \mathbf{i}_4 \to \mathbf{i}_5 \to \mathbf{i}_6$

(ii) Rotational symmetry: $\mathbf{i}_0 \to \mathbf{i}_4 \to \mathbf{i}_5$ and (simultaneously) $\mathbf{i}_2 \to \mathbf{i}_1 \to \mathbf{i}_6$

We could also try to use reflection symmetry, but this isn't a true symmetry as it changes the direction of some of the arrows, and so requires us to carefully change the signs of certain terms: I think trying to use it is more trouble than it's worth.

We can use the cyclic symmetry to choose w.l.o.g. $a = \mathbf{i}_3$; then the rotational symmetry means we need only choose b from one of $\{\mathbf{i}_3, \mathbf{i}_0, \mathbf{i}_2\}$. This reduces the possible value choices to only 21, so there are 63 independent equations to evaluate. The symmetries mean that if (1), (2), and (3) hold for each of these 21 possibilities, they will hold for all 343.

However, there is additional symmetry we can exploit, because we are using only 3 variables: We can examine the special cases where two or more variables are equal, or where the 3 variables are collinear (i.e. lie on a single line or cycle in the Fano plane). In what follows, I will assume that different variables are <u>unequal</u> unless stated otherwise.

We will need the following 4 very simple lemmas:

L1: $xy + yx = 0$ … true since unequal **i**'s are anticommutative.

L2: $y(xy) = (yx)y = -(xy)y = -y(yx) = x$

This follows from a *local* symmetry of the Fano plane, namely that for each two elements x and y, there is a unique line (cycle) either $x \to y \to z$ or $y \to x \to z$. Thus $y(xy) = \pm yz = x$ (the ± depending on cycle direction, but this is factored in twice so it cancels). Similarly for the other terms above.

L3: When x,y,z lie along a directed line (cycle) in the Fano plane, $x(yz) = (xy)z = -y(xz) = -(yx)z = -1$

Since the lines (cycles) define multiplication, we have $x(yz) = x(x)$, and $\mathbf{i}^2 = -1$ for each of the **i**'s.

<u>Special case 1: *a*=*b*=*c*</u>

The associator A(a,a,a) = $a(aa)$-$(aa)a$ = $(-1)a$-$(-1)a$ = 0, so all three identities follow trivially.

Special case 2: $a=b$

(1) $a(ac) + a(ac) - (aa + aa)c = -2c + 2c = 0$ (using L2)
(2) $a(ac + ca) - (aa)c - (ac)a = 0a + c - c = 0$ (using L1, L2)
(3) $a(ac) - (ca)a + c(aa) - (aa)c = -c + c - c + c = 0$ (using L2)

Special case 3: $a=c$

(1) $a(ba) + b(aa) - (ab + ba)a = b - b - 0a = 0$ (using L2, L1)
(2) $a(ba + ab) - (ab)a - (aa)b = 0a - b + b = 0$ (using L1, L2)
(3) $a(ba) - (ab)a + a(ba) - (ab)a = b - b + b - b = 0$ (using L2)

Special case 4: $b=c$

(1) $a(bb) + b(ab) - (ab + ba)b = -a + a - 0b = 0$ (using L2, L1)
(2) $a(bb + bb) - (ab)b - (ab)b = -2a + 2a = 0$ (using L2)
(3) $a(bb) - (bb)a + b(ba) - (ab)b = -a + a - a + a = 0$ (using L2)

Special case 5: a,b,c collinear in the Fano plane ($\mp/\pm$ depending on cycle direction)

(1) $a(bc) + b(ac) - (ab + ba)c = \mp 1 \pm 1 - 0c = 0$ (using L3, L1)
(2) $a(bc + cb) - (ab)c - (ac)b = 0a \pm 1 \mp 1 = 0$ (using L1, L3)
(3) $a(bc) - (cb)a + c(ba) - (ab)c = \mp 1 \mp 1 \pm 1 \pm 1 = 0$ (using L3)

Remaining case: a,b,c all different, and not collinear in the Fano plane:

Note that when a,b,c are not collinear, $bc \neq \pm a$, and since unequal **i**'s are *anticommutative* we have that $c(ba) = -c(ab) = -(ba)c = (ab)c$. Thus:

(3) $a(bc) - (cb)a + c(ba) - (ab)c = a(bc) - a(bc) + (ab)c - (ab)c = 0$ (using anticommutativity)

Also, (1) and (2) simplify:

(1) $A(a,b,c) + A(b,a,c) = 0 \quad \Leftrightarrow \quad a(bc) + b(ac) = 0$ (using L1)
(2) $A(a,b,c) + A(a,c,b) = 0 \quad \Leftrightarrow \quad -(ab)c + b(ac) = 0$ (using L1 and anticommutativity)

Combining them gives:

(4) $a(bc) + (ab)c = 0$

…and actually this is all we require: Noting that the equation is unchanged by *permuting* a,b,c, we have $(ab)c = -c(ab) = (ca)b = b(ac)$ (using anticommutativity and a permutation of (4)), and substituting this result back into (4) recovers (1) and (2).

Of the 21 independent sets of values for [a,b,c] that global symmetry requires us to check, 13 of these fall into the special cases above, and so the antisymmetry of the associator is already demonstrated in those cases. We need only manually confirm that equation (4) holds for the remaining 8 cases:

[a, b, c]	Evaluation of LHS of (4) (should be 0)
[$\mathbf{i}_3$, $\mathbf{i}_0$, $\mathbf{i}_2$]	$\mathbf{i}_3(\mathbf{i}_0\mathbf{i}_2) + (\mathbf{i}_3\mathbf{i}_0)\mathbf{i}_2 = \mathbf{i}_3\mathbf{i}_6 + \mathbf{i}_1\mathbf{i}_2 = -\mathbf{i}_4 + \mathbf{i}_4 = 0$
[$\mathbf{i}_3$, $\mathbf{i}_0$, $\mathbf{i}_4$]	$\mathbf{i}_3(\mathbf{i}_0\mathbf{i}_4) + (\mathbf{i}_3\mathbf{i}_0)\mathbf{i}_4 = \mathbf{i}_3\mathbf{i}_5 + \mathbf{i}_1\mathbf{i}_4 = \mathbf{i}_2 - \mathbf{i}_2 = 0$
[$\mathbf{i}_3$, $\mathbf{i}_0$, $\mathbf{i}_5$]	$\mathbf{i}_3(\mathbf{i}_0\mathbf{i}_5) + (\mathbf{i}_3\mathbf{i}_0)\mathbf{i}_5 = -\mathbf{i}_3\mathbf{i}_4 + \mathbf{i}_1\mathbf{i}_5 = -\mathbf{i}_6 + \mathbf{i}_6 = 0$
[$\mathbf{i}_3$, $\mathbf{i}_0$, $\mathbf{i}_6$]	$\mathbf{i}_3(\mathbf{i}_0\mathbf{i}_6) + (\mathbf{i}_3\mathbf{i}_0)\mathbf{i}_6 = -\mathbf{i}_3\mathbf{i}_2 + \mathbf{i}_1\mathbf{i}_6 = \mathbf{i}_5 - \mathbf{i}_5 = 0$
[$\mathbf{i}_3$, $\mathbf{i}_2$, $\mathbf{i}_0$]	$\mathbf{i}_3(\mathbf{i}_2\mathbf{i}_0) + (\mathbf{i}_3\mathbf{i}_2)\mathbf{i}_0 = -\mathbf{i}_3\mathbf{i}_6 - \mathbf{i}_5\mathbf{i}_0 = \mathbf{i}_4 - \mathbf{i}_4 = 0$
[$\mathbf{i}_3$, $\mathbf{i}_2$, $\mathbf{i}_1$]	$\mathbf{i}_3(\mathbf{i}_2\mathbf{i}_1) + (\mathbf{i}_3\mathbf{i}_2)\mathbf{i}_1 = -\mathbf{i}_3\mathbf{i}_4 - \mathbf{i}_5\mathbf{i}_1 = -\mathbf{i}_6 + \mathbf{i}_6 = 0$
[$\mathbf{i}_3$, $\mathbf{i}_2$, $\mathbf{i}_4$]	$\mathbf{i}_3(\mathbf{i}_2\mathbf{i}_4) + (\mathbf{i}_3\mathbf{i}_2)\mathbf{i}_4 = \mathbf{i}_3\mathbf{i}_1 - \mathbf{i}_5\mathbf{i}_4 = -\mathbf{i}_0 + \mathbf{i}_0 = 0$
[$\mathbf{i}_3$, $\mathbf{i}_2$, $\mathbf{i}_6$]	$\mathbf{i}_3(\mathbf{i}_2\mathbf{i}_6) + (\mathbf{i}_3\mathbf{i}_2)\mathbf{i}_6 = \mathbf{i}_3\mathbf{i}_0 - \mathbf{i}_5\mathbf{i}_6 = \mathbf{i}_1 - \mathbf{i}_1 = 0$

…and it does. This establishes the antisymmetry of the associator (in all its arguments) when applied to the individual generating elements (the **i**'s).

■ (part 1)

Part 2 asks us to generalize this result to *all* elements of the octonion algebra, and to show that this implies $a(ab) = (aa)b$ for arbitrary elements a,b.

I'll show the 2nd statement first.

Assume the associator $A(a,b,c) = a(bc)-(ab)c$ is antisymmmetric in each pair of arguments.

Then in particular $A(a,c,b) + A(c,a,b) = 0$.

Let $c=a$. Substituting $\Rightarrow$ $A(a,a,b) + A(a,a,b) = 0$

$\Rightarrow$ $0 = A(a,a,b) = a(ab)-(aa)b$

$\Rightarrow$ $a(ab) = (aa)b$ as required.

■ (part 2b)

It remains to show that the associator is antisymmetric when its arguments are *arbitrary* elements of the octonion algebra; that is, for the arbitrary variables:

$$a = a_r + a_0\mathbf{i}_0 + a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3 + a_4\mathbf{i}_4 + a_5\mathbf{i}_5 + a_6\mathbf{i}_6$$

$$b = b_r + b_0\mathbf{i}_0 + b_1\mathbf{i}_1 + b_2\mathbf{i}_2 + b_3\mathbf{i}_3 + b_4\mathbf{i}_4 + b_5\mathbf{i}_5 + b_6\mathbf{i}_6 \qquad a_X, b_X, c_X \in \mathbb{R}$$

$$c = c_r + c_0\mathbf{i}_0 + c_1\mathbf{i}_1 + c_2\mathbf{i}_2 + c_3\mathbf{i}_3 + c_4\mathbf{i}_4 + c_5\mathbf{i}_5 + c_6\mathbf{i}_6$$

Consider $A(a,b,c) + A(b,a,c)$

$= a(bc) - (ab)c + b(ac) - (ba)c$.

Since the distributive law applies in the octonion algebra, we have

$= (a_r + a_0\mathbf{i}_0 + \ldots + a_6\mathbf{i}_6)((b_r + b_0\mathbf{i}_0 + \ldots + b_6\mathbf{i}_6)(c_r + c_0\mathbf{i}_0 + \ldots + c_6\mathbf{i}_6)) - \ldots$

$= [a_rb_rc_r + \ldots + a_i\mathbf{i}_i(b_j\mathbf{i}_j)(c_k\mathbf{i}_k)) + \ldots + a_6\mathbf{i}_6(b_6\mathbf{i}_6)(c_6\mathbf{i}_6))] - \ldots$

Here i,j,k each run through the subscripts r, 0, 1, … ,6. Take "$\mathbf{i}_r$" to simply mean "1".

$= [a_rb_rc_r + \ldots + a_ib_jc_k(\mathbf{i}_i(\mathbf{i}_j\mathbf{i}_k)) + \ldots + a_6b_6c_6(\mathbf{i}_6(\mathbf{i}_6\mathbf{i}_6))] - \ldots$

Grouping terms with the same i,j,k together:

$$= \sum a_ib_jc_k\big(A(\mathbf{i}_i, \mathbf{i}_j, \mathbf{i}_k) + A(\mathbf{i}_j, \mathbf{i}_i, \mathbf{i}_k)\big)$$

We have already shown in part 1 that the bracketed term is zero for all $i,j,k \in \{0, 1, \ldots, 6\}$. When one (or more) arguments to the associator A() is the real value 1, however:

$$A(1,b,c) = 1(bc)-(1b)c = bc - bc = 0$$

$$A(a,1,c) = a(1c)-(a\times1)c = ac - ac = 0$$

$$A(a,b,1) = a(b\times1)-(ab)\times1 = ab - ab = 0$$

…and so the bracketed term is zero for every combination of $i,j,k \in \{r, 0, 1, \ldots, 6\}$

Hence $A(a,b,c) + A(b,a,c) = 0$ for *arbitrary* elements of the octonion algebra a,b,c; i.e. the associator is antisymmetric in its first two arguments.

Proof of antisymmetry in the other two pairs of arguments follows analogously.

Therefore, the associator $A(a,b,c) = a(bc)-(ab)c$ is antisymmmetric in each pair of arguments, when its domain is the set of all elements of the octonion algebra.

This completes the exercise!

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