Exercise [15.05]

This exercise asks us to show that the Clifford bundle S^3 (a fibre bundle with base space S^2 and fibre S^1) can be regarded as S^2 × {spinorial tangent vectors on S^2}. We *already* have the map from S^3 onto S^2, namely the canonical projection, so it remains to find a continuous map from S^3 to unit tangent vectors on S^2 such that (a) the *position* of the vector on S^2 is given by the canonical projection, and (b) as we traverse the fibre associated with a point on S^2, we rotate through all possible unit tangent vectors at that point. (In fact it will turn out that going once around the fibre rotates *twice* around all such tangent vectors - which is why the points on S^3 represent *spinorial* tangent vectors… but that will be clearer when we find the actual map).

Now, Penrose gives us a hint for this exercise: The 4-dimensional vector field:

$$-v\frac{\partial}{\partial u} \quad + \quad u\frac{\partial}{\partial v} \quad - \quad y\frac{\partial}{\partial x} \quad + \quad x\frac{\partial}{\partial y}$$

which can also be written as $\begin{bmatrix}-v & u & -y & x\end{bmatrix}^T$ if we take (u,v,x,y) as a 4-dimensional Cartesian coordinate system.

Why is he giving us a *4-dimensional* vector field, when what we want is a map to vectors on the *2-dimensional* surface S^2? The answer is that the Clifford bundle S^3 is (or can be) embedded in 4-dimensional Euclidean space (or alternatively 2-dimensional complex space, which has 4 real dimensions), so vectors on S^3 can be written as vectors in this 4-space; and furthermore because the canonical projection from S^3 to S^2 can be used to project *vectors* on S^3 to vectors on S^2, as well as projecting points. So we might expect that the vector field he's given us provides the correct solution once we project it onto S^2.

At this point, we need the nitty gritty details of that projection. We define S^3 as the unit sphere in 4-dimensions, according to the equation:

$$u^2 \; + \; v^2 \; + \; x^2 \; + \; y^2 \quad = \quad 1 \tag{1}$$

We then define the complex numbers: $w \; = \; u + iv$ and $z \; = \; x + iy$. Their ratio $P \; = \; \dfrac{w}{z}$ is a point on $\mathbb{C} \cup \{\infty\}$, so we can map it onto the Riemann sphere S^2 using stereographic projection. In coordinate terms, if we take $P \; \equiv \; g + ih$ then $\left(\text{using } \dfrac{w}{z} = \dfrac{w\bar{z}}{|z|^2} \right)$:

$$g \quad = \quad \frac{ux + vy}{x^2 + y^2} \qquad\qquad h \quad = \quad \frac{vx - uy}{x^2 + y^2} \tag{2}$$

Stereographic projection onto S^2 (embedded in $\mathbb{R}^3$) then gives us:

$$x' = \frac{2g}{1+|P|^2} \qquad y' = \frac{2h}{1+|P|^2} \qquad z' = \frac{1-|P|^2}{1+|P|^2}$$

Or in terms of the original coordinates:

$$x' = \frac{2ux+2vy}{u^2+v^2+x^2+y^2} \qquad y' = \frac{2vx-2uy}{u^2+v^2+x^2+y^2} \qquad z' = \frac{-u^2-v^2+x^2+y^2}{u^2+v^2+x^2+y^2} \tag{3}$$

We need to project vectors also. Let $\begin{bmatrix} a & b & c & d \end{bmatrix}^T$ be a vector in (u,v,x,y) coordinate space, at the point (u,v,x,y). Then the projected vector at (g,h) on the complex (P-)plane has components:

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

(This can be interpreted as $r+is$, since a vector on the complex plane is just a complex number). So:

$$\begin{bmatrix} r \\ s \end{bmatrix} = \frac{1}{x^2+y^2} \begin{bmatrix} ax+by+cu+dv-\dfrac{2c\left(ux^2+vxy\right)+2d\left(uxy+vy^2\right)}{x^2+y^2} \\ -ay+bx+cv-du-\dfrac{2c\left(vx^2-uxy\right)+2d\left(vxy-uy^2\right)}{x^2+y^2} \end{bmatrix} \tag{4}$$

Similarly the projection of $\begin{bmatrix} a & b & c & d \end{bmatrix}^T$ onto S^2 (embedded in $\mathbb{R}^3$) has components in the $\mathbb{R}^3$ (x',y',z') coordinate system of:

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial u} & \frac{\partial x'}{\partial v} & \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial u} & \frac{\partial y'}{\partial v} & \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \\ \frac{\partial z'}{\partial u} & \frac{\partial z'}{\partial v} & \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Evaluating and simplifying by substituting (1) yields:

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} 2a\left(-x+2x^3-2uvy+2v^2x+2xy^2\right)+2b\left(y-2uvx-2v^2y\right)+2c\left(u-2ux^2-2vxy\right)+2d\left(v-2uxy-2vy^2\right) \\ 2a\left(y-2y^3-2uvx-2v^2y-2x^2y\right)+2b\left(x+2uvy-2v^2x\right)+2c\left(v-2vx^2+2uxy\right)+2d\left(-u-2vxy+2uy^2\right) \\ -4\left(au+bv\right)\left(x^2+y^2\right)+4\left(cx+dy\right)\left(u^2+v^2\right) \end{bmatrix} \tag{5}$$

So, now we can consider Penrose's hint vector, $\begin{bmatrix}-v & u & -y & x\end{bmatrix}^T$. First we can confirm that it *is* a vector on S^3 (provided we're at a point on S^3, i.e. (1) is satisfied), because it is perpendicular to the radial vector $\begin{bmatrix}u & v & x & y\end{bmatrix}^T$: The dot product between the two is zero.

Now if we project this vector onto the complex (P-)plane using (4), we get… the zero vector (everywhere). The same happens if we use (5) to project onto S^2 (of course). So the "hint" Penrose provided is not what we might have expected. (Actually, I suspect that it was probably supposed to be, but he wrote down the wrong vector by mistake). Still, it has at least given us the idea of trying to first find *vectors* on S^3 and then projecting them onto S^2 (rather than somehow trying to define a map from points on S^3 to vectors on S^2 directly). It's always possible to do this, since *any* vector on S^2 can be found as a projection of some vector on S^3. That leaves the question of, what vector?

Penrose's vector projects to zero, which means it's tangent to the fibre: travelling in that direction on S^3 leaves you stationary in the same spot on S^2 (after projection). We need a vector perpendicular to that one, and also perpendicular to the radial vector (to ensure it's tangent to S^3 in the embedding 4-d space). Noting that Penrose's vector is just a rearrangement of the coordinate variables, *by inspection* either of the two vectors $\mathbf{v_1}$ or $\mathbf{v_2}$ below fit the bill:

$$\textbf{radial} = \begin{bmatrix} u \\ v \\ x \\ y \end{bmatrix} \qquad \begin{matrix}\textbf{fibre} \\ \textbf{tangent}\end{matrix} = \begin{bmatrix} -v \\ u \\ -y \\ x \end{bmatrix} \qquad \mathbf{v_1} = \begin{bmatrix} y \\ x \\ -v \\ -u \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} x \\ -y \\ -u \\ v \end{bmatrix}$$

In fact, these vectors are all mutually perpendicular (dot products are zero between any pair), and the last three are unit vectors spanning the tangent space everywhere on S^3. Since the fibre tangent vector projects to zero, the vector we need as a solution is therefore some linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$ at each point on S^3. This linear combination *might* turn out to be some arbitrarily complex function of position on S^3… but if you suspect not, you'd be right! In fact, either $\mathbf{v_1}$ or $\mathbf{v_2}$ on their own will give the solution we are looking for. I'll demonstrate this using $\mathbf{v_1}$:

First, project $\mathbf{v_1}$ onto S^2 using (5) and simplifying, to get:

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} 4(xy-uv) \\ 2-4(v^2+y^2) \\ -4(vx+uy) \end{bmatrix} \tag{6}$$

After some further simplification and substitution of (1) this yields: $a'^2+b'^2+c'^2 = 4$

So, the projection of $\frac{1}{2}\mathbf{v_1}$ onto S^2 is always a unit vector.

It remains to show that the projection of $\frac{1}{2}\mathbf{v_1}$ onto S^2 rotates around as you travel along the (S^1) fibre in S^3 corresponding to any particular point of S^2. This is easiest to demonstrate using the vector $\begin{bmatrix} r & s \end{bmatrix}^T$ on the complex (P-)plane, rather than on the Riemann sphere itself. Since the mapping between the two is conformal, a vector rotating on the plane is also rotating through the same angle on the Riemann sphere.

If $(w_0, z_0) \in \mathbb{C}\times\mathbb{C}$ is some point on S^3 (i.e. $|w_0|^2+|z_0|^2=1$), then the projection to the Riemann sphere S^2 depends only on the ratio $\dfrac{w_0}{z_0}$. Some other point (w,z) will have the same ratio iff $(w,z)=(kw_0,kz_0),\ k\in\mathbb{C}\setminus\{0\}$. However for this point to *also* be on S^3 we require $|k|=1$, and so the complete set of points composing the fibre containing (w_0,z_0) is given by the set $\left\{\left(w_0 e^{i(\theta-\mathrm{Arg}(z_0))},\ |z_0|e^{i\theta}\right):\ \theta\in[0,2\pi)\ \right\}$. Position along the fibre is then parameterised by θ.

Now using (4) to calculate the projection of $\frac{1}{2}\mathbf{v_1}$ onto the complex (P-)plane yields the vector:

$$\begin{bmatrix} r \\ s \end{bmatrix} = \frac{1}{2\left(x^2+y^2\right)^2}\begin{bmatrix} 2xy \\ x^2-y^2 \end{bmatrix} \tag{7}$$

This vector exists at the point $P \equiv g+ih$ as given by (2). To see how this vector changes as we move along the fibre associated with this point, simply set $z = |z_0|e^{i\theta}$ as above, yielding $x=|z_0|\cos\theta,\quad y=|z_0|\sin\theta$. Substituting and applying the double-angle trig. identities gives:

$$\begin{bmatrix} r \\ s \end{bmatrix} = \frac{1}{2|z_0|^2}\begin{bmatrix} \sin 2\theta \\ \cos 2\theta \end{bmatrix}$$

Which makes it immediately clear that as you make one full rotation around the fibre S^1, the tangent vector on S^2 projected from $\frac{1}{2}\mathbf{v_1}$ rotates through *two* full rotations. The points (w, z) and $(-w, -z)$ in S^3 map to the same tangent vector on S^2; the mapping becomes one-to-one iff you map to *spinorial* tangent vectors onto S^2 instead.

(Note that there's no explicit notation for such vectors in component terms, as far as I'm aware; you just have to imagine regular vectors that exist in either one of two "layers" at each point, joined so that going around twice rotates once through each "layer". Figure 8.1 on p. 135 illustrates something similar albeit with infinitely many "layers"; imagine cutting off this spiral after just two rotations then joining the severed ends together, and you'll have the right image.)

The only thing remaining is to check that the above result still holds at $P = \infty$, i.e. when $z = 0$ and $w = e^{i\theta}$ (since the above result isn't valid there). This gives $u = \cos\theta,\ v = \sin\theta,\ x = 0,\ y = 0$. This special case is most easily done by substituting directly into (6) to get the projected vector on S^2 (embedded in $\mathbb{R}^3$). In Cartesian coordinates this vector is:

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} -\sin 2\theta \\ \cos 2\theta \\ 0 \end{bmatrix}$$

…after simplification, again using double-angle trig. identities. Clearly this vector also rotates through two full rotations as you travel once around the fibre. Thus $P = \infty$ is no different to any other point on the Riemann sphere in this respect, and so the exercise is solved: Equations (3) and (6) jointly provide the required continuous map from S^3 simultaneously onto both S^2 and onto the unit tangent space at each correspondingly mapped point of S^2 (respectively), with an exact one-to-one mapping existing if we map to the unit *spinorial* tangent space on S^2, instead.