Exercise [15.04]

The argument can be carried out in a general way for the number space "$\mathbb{X}$", where $\mathbb{X}$ represents either the complex numbers, the quaternions, or the octonions. Let d denote the dimension of $\mathbb{X}$: 2, 4, and 8, respectively.

First we must note that the parameterization of the set of "complex lines" by the ratio $A{:}B$ using the equation (from page 334):

(1) $$Aw + Bz = 0$$

can (in the case of both complex numbers and quaternions) equivalently be expressed as*:

(2) $$\mathcal{M} = \left\{ w = Cz : C \in \mathbb{X} \right\} \cup \left\{ z = C^{-1}w : C^{-1} \in \mathbb{X} \right\}$$

By taking $C = -A^{-1}B$. The elements of the two constituent sets above overlap when parameters C and C^{-1} are reciprocals of each other $\left(C \neq 0,\ C^{-1} \neq 0\right)$; but we also include the two special cases $C = 0$ and $C^{-1} = 0$ in the (overall) set $\mathcal{M}$. (In the latter case, treat "C^{-1}" as an independent symbol; we could just as well have used "D" instead).

Thus, the elements of $\mathcal{M}$ form the points of a manifold, coverable by the two coordinate patches $C \in \mathbb{X}$ and $C^{-1} \in \mathbb{X}$ (which overlap everywhere except at the two points $C = 0$ and $C^{-1} = 0$).

Topologically, this manifold $\mathcal{M}$ is S^d (the d-sphere). The argument showing this for the higher dimension cases (quaternions and octonions) is exactly the same as for the complex plane - you can simply use stereographic projection as in diagram 8.7 on page 144 – but with S^d and an $\mathbb{X}$-plane in place of the Riemann sphere and complex plane. I won't repeat the details here.

Note that the two equations above are <u>not</u> equivalent in the case of octonions, because associativity fails, so $A^{-1}(Bz) \neq (A^{-1}B)z$. This means we can no longer combine the two parameters A and B into a single parameter C, in any fashion. Thus, we <u>must</u> start out using (2) rather than (1) if we wish the argument to work for octonions.

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* Strictly speaking this ought to be written: $\mathcal{M} = \{\{ (w,z): w = Cz, (w,z) \in \mathbb{X}\times\mathbb{X} \}: C \in \mathbb{X}\} \cup \{\{ (w,z): z = Dw, (w,z) \in \mathbb{X}\times\mathbb{X} \}: D \in \mathbb{X}\}$

We proceed by noting that each element of $\mathcal{M}$ represents an "$\mathbb{X}$ line" – a linear relation between $w \in \mathbb{X}$ and $z \in \mathbb{X}$; that is, each element represents a *d-dimensional linear subspace* of the embedding $2d$-dimensional vector space $\mathbb{X}\times\mathbb{X}$. For a given equation $s \in \mathcal{M}$ (defined by some particular choice of C or C^{-1}), the corresponding linear subspace is $\{(w,z) : (w,z) \text{ satisfies } s\} \subset \mathbb{X}\times\mathbb{X}$.

(Here "(w,z) satisfies s" means $w = Cz$ or $z = C^{-1}w$ for the given C or C^{-1} that defines s).

Now, the point $(0,0)$ is included in *every* subspace $s \in \mathcal{M}$. However, with the specific exclusion of this point, the subspaces $s \in \mathcal{M}$ are non-intersecting, and also cover the entire space $\mathbb{X}\times\mathbb{X}$ – i.e. for any given point $(w,z) \neq (0,0)$ in $\mathbb{X}\times\mathbb{X}$, there is a unique $s \in \mathcal{M}$ containing that point: It is given by either $C = w.z^{-1}$ or $C^{-1} = z.w^{-1}$.

So, consider the unit $S^{2d\text{-}1}$ sphere $\mathcal{B} \subset \mathbb{X}\times\mathbb{X}$ defined by the equation:

$$|w|^2 + |z|^2 = 1$$

Note that this *is* a sphere in the $2d$ dimensional vector space $\mathbb{X}\times\mathbb{X}$, because for each of our choices of $\mathbb{X}$, the square of the modulus is just the sum of the squares of the (real, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, etc…) components. Clearly this sphere does not include the point $(0,0)$. For each point on this sphere then, there is a unique $s \in \mathcal{M}$ representing the d-dimensional linear subspace in $\mathcal{M}$ that intersects that point. Thus this yields a well-defined function from $\mathcal{B}$ to $\mathcal{M}$.

We are attempting to demonstrate that $\mathcal{B}$ is a fibre bundle over the base space $\mathcal{M}$. We can therefore take the above function as the canonical projection from $\mathcal{B}$ to $\mathcal{M}$. It remains only to show that the "fibres" of this bundle are well defined – i.e. that for each point $s \in \mathcal{M}$, the image of s in $\mathcal{B}$ is a (sub)manifold $\mathcal{V}$, and that these $\mathcal{V}$s all have identical structure (for each $s \in \mathcal{M}$).

Now, we know that the intersection of any sphere $S^{n\text{-}1}$ embedded in an n-dimensional vector space with any m-dimensional linear subspace passing through that sphere's origin is always another sphere $S^{m\text{-}1}$ (provided that $1 < m < n$).

(To see this, note that the equation of a sphere centered at the origin is unaffected by coordinate transformations that are rotations around the origin, and that you can use such rotations to map any linear subspace through the origin to the particular subspace that uses only the first m coordinates and has the others set zero; at which point the equation of the $(n-1)$-sphere reduces to that of a $(m-1)$-sphere).

Therefore the intersection of each linear subspace $s \in \mathcal{M}$ in $\mathbb{X}\times\mathbb{X}$ with the sphere $\mathcal{B}$ (i.e. the image of s in $\mathcal{B}$) forms a sphere $S^{d\text{-}1}$ that is a submanifold of $\mathcal{B}$. Thus the "fibre" space $\mathcal{V}$ is well defined, and has the structure of $S^{d\text{-}1}$.

Therefore $\mathcal{B}$ can be regarded as a $\mathcal{V}$-bundle over $\mathcal{M}$.

$\mathcal{B}$ is the sphere $S^{2d\text{-}1}$.

$\mathcal{M}$ is the sphere S^{d}.

$\mathcal{V}$ is the sphere $S^{d\text{-}1}$.

Substituting d = 2, 4, and 8 for each of the possible choices of $\mathbb{X}$ (complex numbers, quaternions, and octonions respectively) yields the following results:

S^3 can be regarded as an S^1 bundle over S^2.

S^7 can be regarded as an S^3 bundle over S^4.

S^{15} can be regarded as an S^7 bundle over S^8.

Note that the argument fails for any higher dimension, because (as stated on page 202) the corresponding higher-dimensional algebras don't support division, which is required to relate C to C^{-1} (thus joining the two coordinate patches on $\mathcal{M}$), and also to ensure that there's a unique $C = w.z^{-1}$ or $C^{-1} = z.w^{-1}$ for each point (w,z) on $\mathcal{B}$, which is required for defining the canonical projection from $\mathcal{B}$ to $\mathcal{M}$. (Note also that in higher dimensional algebras the squared modulus/norm is no longer the sum of the squares of the components, so the equation for a sphere used to define $\mathcal{B}$ above would need to be written out in component form instead; we can do that, but it's superfluous to the argument in any case).