Double Poisson brackets

It is, by definition:

$$\{\Phi,\{\Theta,\Psi\}\}=1/2\,S^{ab}\nabla_a\Phi\nabla_b(1/2\,S^{cd}\nabla_c\Theta\nabla_d\Psi)=$$
$$=1/4\,S^{ab}\nabla_a\Phi\left[\nabla_c\Theta\nabla_d\Psi\nabla_b S^{cd}+S^{cd}\nabla_b\nabla_c\Theta\nabla_d\Psi+S^{cd}\nabla_c\Theta\nabla_b\nabla_d\Psi\right] \quad \text{(I)}$$

In the following, the connection coefficients Γ^c_{ab} will not be assumed symmetric under exchange $a\leftrightarrow b$, i.e. a torsion may be present (see exercise 14.7).
I'll also write ∂_a as an abbreviation of coordinate derivative $\partial/\partial x^a$.

To find the covariant derivative of S^{ab} , we can compute:

$$\nabla_a(S^{bc}S_{cd})=(\nabla_a S^{bc})S_{cd}+S^{bc}(\nabla_a S_{cd})=(\nabla_a S^{bc})S_{cd}+S^{bc}(\partial_a S_{cd}-\Gamma^s_{ac}S_{sd}-\Gamma^s_{ad}S_{cs})=$$
$$=(\nabla_a S^{bc})S_{cd}+S^{bc}\partial_a S_{cd}-\Gamma^s_{ac}S_{sd}S^{bc}-\Gamma^b_{ad}$$

and being

$$\nabla_a(\delta^b_d)=\partial_a\delta^b_d+\Gamma^b_{as}\delta^s_d-\Gamma^s_{ad}\delta^b_s=\Gamma^b_{ad}-\Gamma^b_{ad}=0$$

it is

$$(\nabla_a S^{bc})S_{cd}=-S^{bc}\partial_a S_{cd}+\Gamma^s_{ac}S_{sd}S^{bc}+\Gamma^b_{ad}$$

$$(\nabla_a S^{bc})S_{cd}S^{df}=-S^{bc}S^{df}\partial_a S_{cd}+\Gamma^s_{ac}S^{df}S_{sd}S^{bc}+\Gamma^b_{ad}S^{df}$$

$$\nabla_a S^{bf}=-S^{bc}S^{df}\partial_a S_{cd}+\Gamma^f_{ac}S^{bc}+\Gamma^b_{ad}S^{df} \quad \text{(II)}$$

It is also:

$$\nabla_a\nabla_b\Phi=\partial_a\partial_b\Phi-\Gamma^s_{ab}\nabla_s\Phi \quad \text{(III)}$$

and substituting (II) and (III) into (I) , the expression into square brackets becomes:

$$\partial_c\Theta\partial_d\Psi(-S^{cp}S^{qd}\partial_b S_{pq}+\Gamma^d_{bp}S^{cp}+\Gamma^c_{bp}S^{pd})+$$
$$+S^{cd}\partial_b\partial_c\Theta\,\partial_d\Psi+S^{cd}\partial_c\Theta\,\partial_b\partial_d\Psi+$$
$$-S^{cd}\Phi\,\Gamma^s_{bc}\partial_s\Theta\,\partial_d\Psi-S^{cd}\partial_c\Theta\,\Gamma^s_{bd}\partial_s\Psi$$

The second term in first line and the second in last line cancel each other, and same happens to third term of first line and first of third line, so that finally:

$$\{\Phi,\{\Theta,\Psi\}\}=1/4\,S^{ab}\partial_a\Phi\left[-S^{cp}S^{qd}\partial_b S_{pq}\partial_c\Theta\,\partial_d\Psi+S^{cd}\partial_b\partial_c\Theta\,\partial_d\Psi+S^{cd}\partial_c\Theta\,\partial_b\partial_d\Psi\right]$$

This expression is manifestly independent from the connection.