

The point is that covariant derivative (or parallel transport, or connection) is not uniquely defined.

Everyone is free to define its own covariant derivative ∇T of a tensor T as he wants, provided that definition satisfies the formal properties of derivative, listed in chapter 14.3:

- a) $\nabla(T+U) = \nabla T + \nabla U$ additivity
- b) $\nabla(T \cdot U) = (\nabla T) \cdot U + T \cdot (\nabla U)$ Leibniz's rule
- c) $\nabla \Phi = \frac{\partial \Phi}{\partial x^a}$ when Φ is a scalar

For instance I can define my own "Roberto's derivative" for a vector and a covector according to the following definitions:

$$\begin{aligned} \overset{R}{\nabla} T &= \overset{R}{\nabla}_b T^a = \frac{\partial T^a}{\partial x^b} + R_{bc}^a T^c \\ \overset{R}{\nabla} U &= \overset{R}{\nabla}_b U_a = \frac{\partial U_a}{\partial x^b} - R_{ba}^c U_c \end{aligned}$$

where R_{bc}^a is my favourite three-indices smooth function of the point. This satisfies a) for vectors:

$$\overset{R}{\nabla}(T+U) = \frac{\partial(T^a+U^a)}{\partial x^b} + R_{bc}^a(T^c+U^c) = \left(\frac{\partial T^a}{\partial x^b} + R_{bc}^a T^c\right) + \left(\frac{\partial U^a}{\partial x^b} + R_{bc}^a U^c\right) = \overset{R}{\nabla} T + \overset{R}{\nabla} U$$

and same for covectors. It satisfies also b):

$$\begin{aligned} (\overset{R}{\nabla} T) \cdot U + T \cdot (\overset{R}{\nabla} U) &= \left(\frac{\partial T^a}{\partial x^b} + R_{bc}^a T^c\right) U_a + T^a \left(\frac{\partial U_a}{\partial x^b} - R_{ba}^c U_c\right) = \\ &= \left(\frac{\partial T^a}{\partial x^b} U_a + T^a \frac{\partial U_a}{\partial x^b}\right) + (R_{bc}^a T^c U_a - T^a R_{ba}^c U_c) = \left(\frac{\partial(T^a U_a)}{\partial x^b}\right) + 0 = \overset{R}{\nabla}(T \cdot U) \end{aligned}$$

Using Leibniz rule (b) one could then extend the definition to any tensor.

You can select your favourite three-indices smooth function of the point D_{bc}^a and define your own "DimBulb's derivative":

$$\begin{aligned} \overset{D}{\nabla} T &= \overset{D}{\nabla}_b T^a = \frac{\partial T^a}{\partial x^b} + D_{bc}^a T^c \\ \overset{D}{\nabla} U &= \overset{D}{\nabla}_b U_a = \frac{\partial U_a}{\partial x^b} - D_{ba}^c U_c \end{aligned}$$

that is as good as mine.

Note that the difference between my and your derivative is

$$\overset{R}{\nabla} T - \overset{D}{\nabla} T = (R_{bc}^a - D_{bc}^a) T^c = \Gamma_{bc}^a T^c$$

Exercise 14.5, first part, asks to prove that the difference between **any two possible definitions** of covariant derivative of a vector T has this form $\Gamma_{bc}^a T^c$.