Exercise 10.1

For the purposes of this exercise I will denote the complex number $z = x + iy$ by the ordered pair (x, y)

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are any two complex numbers then we can define addition and multiplication of complex numbers as follows:

$z_1 + z_2$ is $(x_1 + x_2, y_1 + y_2)$ (1)

$z_1 z_2$ is $(x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$ (2)

If we take the situation where $z_1 = (-1, 0)$ then using (2) we can write:

$(-1, 0) \cdot z_2 = (-x_2, -y_2) = -x_2 - iy_2 = -z_2$

We can now define subtraction as follows:

$z_1 - z_2 = z_1 + (-z_2) = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2)$

and division as follows:

Consider $z_3 = z_1 z_2$, then using (2) we can write

$(x_3, y_3) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

and so $x_3 = x_1 x_2 - y_1 y_2$ and $y_3 = x_1 y_2 + x_2 y_1$

If we now solve these equations for x_2 and y_2 we obtain

$$x_2 = \frac{x_1 x_3 + y_1 y_3}{x_1^2 + y_1^2} \text{ and } y_2 = \frac{x_1 y_3 - y_1 x_3}{x_1^2 + y_1^2}$$

This result allows us to define division as

$$z_2 = (x_2, y_2) = \frac{z_3}{z_1} = \left(\frac{x_1 x_3 + y_1 y_3}{x_1^2 + y_1^2}, \frac{x_1 y_3 - y_1 x_3}{x_1^2 + y_1^2} \right)$$