Exercise [12.16] (amended)

[12.16] First assertion:

$\alpha_{[rs}\alpha_{u]v} = 0 \quad \Leftrightarrow \quad \alpha_{rs} = \gamma_{[r}\delta_{s]}$ (i.e. α_{rs} is simple)

Proof "⇒"

Choose vectors x^ν and z^μ such that

$$z^\mu x^\nu \alpha_{\mu\nu} = 2 \tag{1}$$

(This is always possible if $\alpha \neq 0$, because for the choice $x^\nu = \delta^{\nu k}$ and $z^\mu = \delta^{\mu m}$ the sum reduces to a single component α_{mk}. Choosing *m, k* such that a nonzero $\alpha_{mk} \neq 0$ is selected, and normalizing x^ν and/or z^μ appropriately yields eq.(1).)

We can then define:

$$\gamma_r = x^\nu \alpha_{r\nu} \qquad\qquad \delta_s = z^\mu \alpha_{\mu s} = -z^\mu \alpha_{s\mu} \tag{2}$$

(the last equation holds because α_{rs} can be assumed to be antisymmetric – cf. Penrose, Chapter 12.4). We then get:

$$2 \cdot \gamma_{[r}\delta_{s]} = (\gamma_r\delta_s - \gamma_s\delta_r) \overset{(2)}{=} x^\nu \alpha_{r\nu} \cdot z^\mu \alpha_{\mu s} - x^\nu \alpha_{s\nu} \cdot z^\mu \alpha_{\mu r} = x^\nu z^\mu (\alpha_{r\nu} \cdot \alpha_{\mu s} - \alpha_{s\nu} \cdot \alpha_{\mu r}) \tag{3}$$

Next we exploit the "⇒" prerequisite, i.e.

$$0 = \alpha_{[rs}\alpha_{\mu]\nu} = \frac{1}{3!}\left(\alpha_{rs} \cdot \alpha_{\mu\nu} + \alpha_{s\mu} \cdot \alpha_{r\nu} + \underline{\alpha_{\mu r} \cdot \alpha_{s\nu}} - \alpha_{sr} \cdot \alpha_{\mu\nu} - \underline{\alpha_{\mu s} \cdot \alpha_{r\nu}} - \alpha_{r\mu} \cdot \alpha_{s\nu}\right) \tag{4}$$

to replace the bracket in eq. (3) (the terms of the bracket are underlined in (4)):

$$2 \cdot \gamma_{[r}\delta_{s]} = \ldots \overset{(4)}{=} x^\nu z^\mu (\alpha_{rs} \cdot \alpha_{\mu\nu} + \alpha_{s\mu} \cdot \alpha_{r\nu} - \alpha_{sr} \cdot \alpha_{\mu\nu} - \alpha_{r\mu} \cdot \alpha_{s\nu})$$

$$= \alpha_{rs} \cdot x^\nu z^\mu \alpha_{\mu\nu} + z^\mu \alpha_{s\mu} \cdot x^\nu \alpha_{r\nu} - \alpha_{sr} \cdot x^\nu z^\mu \alpha_{\mu\nu} - z^\mu \alpha_{r\mu} \cdot x^\nu \alpha_{s\nu}$$

$$\overset{(1)(2)}{=} \alpha_{rs} \cdot 2 + (-\delta_s) \cdot \gamma_r - \alpha_{sr} \cdot 2 - (-\delta_r) \cdot \gamma_s = 2\alpha_{rs} - 2\alpha_{sr} + \delta_r \cdot \gamma_s - \delta_s \cdot \gamma_r$$

$$= 2 \cdot (\alpha_{rs} - \alpha_{sr}) - (\gamma_r \cdot \delta_s - \gamma_s \cdot \delta_r) = 4 \cdot \alpha_{rs} - 2 \cdot \gamma_{[r} \cdot \delta_{s]}$$

$\Rightarrow \quad \gamma_{[r}\delta_{s]} = \alpha_{rs}$ q.e.d.

Proof "$\Leftarrow$"

Let α_{rs} be simple, i.e.

$$\alpha_{rs} = \gamma_{[r}\delta_{s]} = \frac{1}{2}(\gamma_r\delta_s - \gamma_s\delta_r) = \frac{1}{2}(r,s - s,r) \qquad (5)$$

The last term of this equation uses the short form

$$\gamma_r\delta_s = r,s \qquad (6)$$

We then have:

$$2\cdot 3!\,\alpha_{[rs}\alpha_{u]v} = 2(\alpha_{rs} - \alpha_{sr})\cdot\alpha_{uv} + 2(\alpha_{su} - \alpha_{us})\cdot\alpha_{rv} + 2(\alpha_{ur} - \alpha_{ru})\cdot\alpha_{sv}$$

$$= 4\alpha_{rs}\cdot\alpha_{uv} + 4\alpha_{su}\cdot\alpha_{rv} + 4\alpha_{ur}\cdot\alpha_{sv}$$ (α_{rs} is antisymmetric!)

$$\overset{(5)}{=} (r,s - s,r)*(u,v - v,u) + (s,u - u,s)*(r,v - v,r) + (u,r - r,u)*(s,v - v,s)$$

$$= \underline{r,s*u,v} - \underline{r,s*v,u} - \underline{s,r*u,v} + \underline{s,r*v,u} + \ldots$$

$$\ldots + \underline{s,u*r,v} - \underline{s,u*v,r} - \underline{u,s*r,v} + u,s*v,r + \ldots$$

$$\ldots + \underline{u,r*s,v} - u,r*v,s - \underline{r,u*s,v} + \underline{r,u*v,s} = 0$$

In the last line, terms that are underlined in the same colour cancel, taking into account that, for example:

$$r,s*u,v = \gamma_r\delta_s*\gamma_u\delta_v = \gamma_r\delta_v*\gamma_u\delta_s = r,v*u,s$$

(i.e. corresponding indices can be swapped in the products).

The aforementioned swapping of indices is possible because they belong to single factors, i.e. because α_{rs} is assumed to be simple. This is surely the secret why the assertion holds also for higher numbers of indices, i.e. $\alpha_{r\ldots s}$, but I did not succeed to find a formal proof.

[12.16] Second assertion:

If $\psi^{r..tu}$ and $\alpha_{uv...w}$ are dual, then

$\psi^{r..tu}\alpha_{uv...w} = 0 \quad \Leftrightarrow \quad \psi^{r..tu}$ and $\alpha_{uv...w}$ are simple

Proof:

Because ψ and α are dual, we have:

$$\psi^{r...tu}\alpha_{uv...w} = \psi^{r...tu}\varepsilon_{uv...wx...z}\psi^{x...z} \tag{7}$$

Two things can be observed with respect to eq.(7):

1. The right sum (summation convention!) comprises only addends in which all *n* indices *uv…wx…z* are different from each other (otherwise $\varepsilon_{uv...wx...z}$ would be zero). The indices *v…w* are given in advance. Due to the summation, the residual numbers from the set *{1,2,…n}* are distributed to the indices *u,x,…z* in all possible permutations.

2. Hence each addend is proportional to $C\cdot\varepsilon_{12...n}\cdot sign(\pi)$ with C being a nonzero constant and π being the permutation of *(u,x…z)* that corresponds to this addend.

Accordingly, eq. (7) can be continued as follows:

$$\psi^{r...tu}\alpha_{uv...w} = C\varepsilon_{12...n}\sum_{\pi(u,x...z)} sign(\pi)\cdot\psi^{r...t\ \pi(u)}\psi^{\pi(x)...\pi(z)} = C\varepsilon_{12...n}(k+1)!\,\psi^{r...t[u}\psi^{x...z]} \tag{8}$$

(with *k* being the number of indices of $\psi^{r...tu}$).

From eq.(8) it follows that

$$\psi^{r...tu}\alpha_{uv...w} = 0 \;\Leftrightarrow\; \psi^{r...t[u}\psi^{x...z]} = 0 \;\Leftrightarrow\; \psi^{r...tu} \text{ is simple (due to the first assertion).}$$

An analogous reasoning holds for $\alpha_{uv...w}$ (using $\in^{uv...wx...z}$ in eq.(7) to express ψ by α).