

Exercise [12.16]

[12.16] First assertion:

$$\alpha_{[rs}\alpha_{u]v} = 0 \quad \Leftrightarrow \quad \alpha_{rs} = \gamma_{[r}\delta_{s]} \quad (\text{i.e. } \alpha_{rs} \text{ is simple})$$

Proof " \Rightarrow "

Choose a vector x^v such that

$$x^\mu x^v \alpha_{\mu v} = 1 \quad (1)$$

(This is always possible unless α is zero, in which case the assertion is trivial.)

With the definition

$$\gamma_r = x^v \alpha_{rv} \quad \delta_s = x^\mu \alpha_{\mu s} \quad (2)$$

we then get:

$$2 \cdot \gamma_{[r} \delta_{s]} = (\gamma_r \delta_s - \gamma_s \delta_r) \stackrel{(2)}{=} x^v \alpha_{rv} \cdot x^\mu \alpha_{\mu s} - x^v \alpha_{sv} \cdot x^\mu \alpha_{\mu r} = x^v x^\mu (\alpha_{rv} \cdot \alpha_{\mu s} - \alpha_{sv} \cdot \alpha_{\mu r}) \quad (3)$$

Next we exploit the " \Rightarrow " prerequisite, i.e.

$$0 = \alpha_{[rs}\alpha_{\mu]v} = \frac{1}{3!} \left(\alpha_{rs} \cdot \alpha_{\mu v} + \alpha_{s\mu} \cdot \alpha_{rv} + \alpha_{\mu r} \cdot \alpha_{sv} - \alpha_{sr} \cdot \alpha_{\mu v} - \alpha_{\mu s} \cdot \alpha_{rv} - \alpha_{r\mu} \cdot \alpha_{sv} \right) \quad (4)$$

to replace the bracket in eq. (3) (the terms of the bracket are underlined in (4)):

$$\begin{aligned} 2 \cdot \gamma_{[r} \delta_{s]} &= \dots \stackrel{(4)}{=} x^v x^\mu \left(\alpha_{rs} \cdot \alpha_{\mu v} + \alpha_{s\mu} \cdot \alpha_{rv} - \alpha_{sr} \cdot \alpha_{\mu v} - \alpha_{r\mu} \cdot \alpha_{sv} \right) \\ &= \alpha_{rs} \cdot x^v x^\mu \alpha_{\mu v} + x^\mu \alpha_{s\mu} \cdot x^v \alpha_{rv} - \alpha_{sr} \cdot x^v x^\mu \alpha_{\mu v} - x^\mu \alpha_{r\mu} \cdot x^v \alpha_{sv} \\ &\stackrel{(1)(2)}{=} \alpha_{rs} \cdot 1 + \gamma_s \cdot \gamma_r - \alpha_{sr} \cdot 1 - \gamma_r \cdot \gamma_s = \alpha_{rs} - \alpha_{sr} \\ &= 2 \cdot \alpha_{rs} \quad \text{q.e.d.} \end{aligned}$$

The last line holds because α_{rs} can be assumed to be antisymmetric (cf. Penrose, Chapter 12.4).

Proof " \Leftarrow "

Let α_{rs} be simple, i.e.

$$\alpha_{rs} = \gamma_{[r} \delta_{s]} = \frac{1}{2}(\gamma_r \delta_s - \gamma_s \delta_r) = \frac{1}{2}(r, s - s, r) \quad (5)$$

The last term of this equation uses the short form

$$\gamma_r \delta_s = r, s \quad (6)$$

We then have:

$$\begin{aligned} 2 \cdot 3! \alpha_{[rs} \alpha_{u]v} &= 2(\alpha_{rs} - \alpha_{sr}) \cdot \alpha_{uv} + 2(\alpha_{su} - \alpha_{us}) \cdot \alpha_{rv} + 2(\alpha_{ur} - \alpha_{ru}) \cdot \alpha_{sv} \\ &= 4\alpha_{rs} \cdot \alpha_{uv} + 4\alpha_{su} \cdot \alpha_{rv} + 4\alpha_{ur} \cdot \alpha_{sv} \quad (\alpha_{rs} \text{ is antisymmetric!}) \end{aligned}$$

$$\stackrel{(5)}{=} (r, s - s, r) * (u, v - v, u) + (s, u - u, s) * (r, v - v, r) + (u, r - r, u) * (s, v - v, s)$$

$$= \underline{r, s * u, v} - \underline{r, s * v, u} - \underline{s, r * u, v} + \underline{s, r * v, u} + \dots$$

$$\dots + \underline{s, u * r, v} - \underline{s, u * v, r} - \underline{u, s * r, v} + \underline{u, s * v, r} + \dots$$

$$\dots + \underline{u, r * s, v} - \underline{u, r * v, s} - \underline{r, u * s, v} + \underline{r, u * v, s} = 0$$

In the last line, terms that are underlined in the same colour cancel, taking into account that, for example:

$$r, s * u, v = \gamma_r \delta_s * \gamma_u \delta_v = \gamma_r \delta_v * \gamma_u \delta_s = r, v * u, s$$

(i.e. corresponding indices can be swapped in the products).

The aforementioned swapping of indices is possible because they belong to single factors, i.e. because α_{rs} is assumed to be simple. This is surely the secret why the assertion holds also for higher numbers of indices, i.e. $\alpha_{r\dots s}$, but I did not succeed to find a formal proof.

[12.16] Second assertion:

If $\psi^{r..tu}$ and $\alpha_{uv...w}$ are dual, then

$$\psi^{r..tu} \alpha_{uv...w} = 0 \quad \Leftrightarrow \quad \psi^{r..tu} \text{ and } \alpha_{uv...w} \text{ are simple}$$

Proof:

Because ψ and α are dual, we have:

$$\psi^{r..tu} \alpha_{uv...w} = \psi^{r..tu} \varepsilon_{uv...wx...z} \psi^{x...z} \quad (7)$$

Two things can be observed with respect to eq.(7):

1. The right sum (summation convention!) comprises only addends in which all n indices $uv...wx...z$ are different from each other (otherwise $\varepsilon_{uv...wx...z}$ would be zero). The indices $v...w$ are given in advance. Due to the summation, the residual numbers from the set $\{1,2,\dots,n\}$ are distributed to the indices u,x,\dots,z in all possible permutations.
2. Hence each addend is proportional to $C \cdot \varepsilon_{12\dots n} \cdot \text{sign}(\pi)$ with C being a nonzero constant and π being the permutation of (u,x,\dots,z) that corresponds to this addend.

Accordingly, eq. (7) can be continued as follows:

$$\psi^{r..tu} \alpha_{uv...w} = C \varepsilon_{12\dots n} \sum_{\pi(u,x,\dots,z)} \text{sign}(\pi) \cdot \psi^{r..t \pi(u)} \psi^{\pi(x)\dots\pi(z)} = C \varepsilon_{12\dots n} (k+1)! \psi^{r..t[u \psi^{x...z}]} \quad (8)$$

(with k being the number of indices of $\psi^{r..tu}$).

From eq.(8) it follows that

$$\psi^{r..tu} \alpha_{uv...w} = 0 \quad \Leftrightarrow \quad \psi^{r..t[u \psi^{x...z}]} = 0 \quad \Leftrightarrow \quad \psi^{r..tu} \text{ is simple (due to the first assertion).}$$

An analogous reasoning holds for $\alpha_{uv...w}$ (using $\varepsilon^{uv...wx...z}$ in eq.(7) to express ψ by α).