If

$$G = \int_{-\infty}^{\infty} e^{-x^2} dx$$

then also

$$G = \int_{-\infty}^{\infty} e^{-y^2} dy$$

In "exterior calculus speak", $e^{-x^2} dx$ and $e^{-y^2} dy$ are 1-forms. The integrations above are along one dimension. If we make the x axis at a right angle to the y axis, then integration over the entire plane of the 2-form $e^{-x^2} dx \wedge e^{-y^2} dy$ will equal G^2.

$$G^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} dx \wedge e^{-y^2} dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx \wedge dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx \wedge dy$$
$$= \int_{\Re^2} e^{-(x^2+y^2)} dx \wedge dy$$

Where $\int_{\Re^2}$ means integrate over the entire plane.

Now the change to polar coordinates

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \sin^{-1} \left(\frac{y}{\sqrt{x^2+y^2}} \right)$$

To integrate over the plane using these coordinates, we need to change the integration from over the 2-form $dx \wedge dy$ to over the 2-form $dr \wedge d\theta$.

$$x = r \cos \theta$$
$$y = r \sin \theta$$

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta$$
$$dx = \cos \theta dr - r \sin \theta d\theta$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta$$
$$dy = \sin \theta dr + r \cos \theta d\theta$$

$dx \wedge dy = (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta)$
$= r\cos^2\theta dr \wedge d\theta - r\sin^2\theta d\theta \wedge dr$
$= r\left(\cos^2\theta + \sin^2\theta\right) dr \wedge d\theta$
$= r dr \wedge d\theta$

so
$G^2 = \int_{\Re^2} e^{-(x^2+y^2)} dx \wedge dy$
$= \int_{\Re^2} r e^{-r^2} dr \wedge d\theta$
$= \int_0^{2\pi} \int_0^{\infty} r e^{-r^2} dr \wedge d\theta$

making yet another change in 2-form by substituting
$u = -r^2$
$du = \frac{du}{dr} dr = -2r dr$

so
$r dr \wedge d\theta = -\frac{1}{2} du \wedge d\theta$

and when $r \longrightarrow \infty; u \longrightarrow -\infty$

$G^2 = \int_0^{2\pi} \int_0^{-\infty} -\frac{1}{2} e^u du \wedge d\theta$

$= \int_0^{2\pi} -\frac{1}{2}\left(e^{-\infty} - e^0\right) d\theta$

$= \int_0^{2\pi} \frac{1}{2} d\theta = \frac{1}{2}(2\pi - 0)$

$= \pi$

$G = \sqrt{\pi}$

The question that still bugs me is, do the original x and y axis have to be orthogonal? I would think that if they were not, if they crossed at some angle α then
$\int_{\Re^2} e^{-(x^2+y^2)} dx \wedge dy = \sqrt{\pi} sin\alpha$

Certainly, if $\alpha = 0$ then $dx \wedge dy = 0$ so the above equation holds, and at $\alpha = \pi/2$ the equation is true, but my math skills fail me to prove the general case.