Sorry Robin, I don't quite understand how you justify your first line:
$\pounds_\xi \eta = [\xi^a \partial_a, \eta^b \partial_b]$
This is clearly true if you apply $\pounds_\xi \eta$ to a scalar, but it's not obviously true in general; in fact it's a statement of the result we're trying to show, for the particular choice of the coordinate connection ($\nabla = \partial$).
I did it this way:
$\pounds_\xi \eta = [\xi, \eta] = \eta(\xi(.)) - \xi(\eta(.))$
Noting that given any particular connection ∇, the action of $\xi(.)$ on an *arbitrary* tensor ϕ is $\xi(\phi) = \xi^a \nabla_a \phi$, we get:
$[\pounds_\xi \eta]\phi = \xi(\eta^b \nabla_b \phi) - \eta(\xi^a \nabla_a \phi)$
$= \xi^a \nabla_a (\eta^b \nabla_b \phi) - \eta^b \nabla_b (\xi^a \nabla_a \phi)$
$= \xi^a (\nabla_a \eta^b)(\nabla_b \phi) + \xi^a \eta^b (\nabla_a \nabla_b \phi) - \eta^b (\nabla_b \xi^a)(\nabla_a \phi) - \eta^b \xi^a (\nabla_b \nabla_a \phi)$
$= (\nabla_\xi \eta)\phi - (\nabla_\eta \xi)\phi + \xi^a \eta^b (\nabla_a \nabla_b - \nabla_b \nabla_a)\phi$
$= [\nabla_\xi \eta - \nabla_\eta \xi + \xi^a \eta^b (\nabla_a \nabla_b - \nabla_b \nabla_a)]\phi$
But we have assumed our (given) ∇ is torsion-free, so the third term in the brackets vanishes, and therefore:
$\pounds_\xi \eta = \nabla_\xi \eta - \nabla_\eta \xi$
for *all* torsion-free connections ∇.