Exercise [12.08]

First, I would like to repeat the motivation/definition of the "antisymmetrization" operation in view of the wedge product of *p* different 1-forms α, … γ. Basically, we have:

$$\alpha \wedge ... \wedge \gamma = \sum_{(r,...u)\in M^p} \alpha_r ... \gamma_u \, dx^r \wedge ... \wedge dx^u \quad \text{with } M = \{1, 2, ... n\} \tag{1}$$

As Penrose explains in §11.6, the above sum comprises groups of *p!* summands each which have the same – but permuted – indices. The wedge products $dx^r \wedge ... \wedge dx^u$ in these summands are not independent, because they differ at most in a "±" sign. It is then desirable to give all those equivalent terms the same coefficients. This is achieved by adding all their coefficients as they appear in eq. (1) (with appropriate sign) and divide this sum by *p!.* The unique coefficient that results from this procedure is the "antisymmetrized" value $\alpha_{[r} \ldots \gamma_{u]}$. Hence we have:

$$\alpha \wedge ... \wedge \gamma = \sum_{(r,...u)\in M^p} \alpha_{[r} ... \gamma_{u]} \, dx^r \wedge ... \wedge dx^u \tag{2}$$

with
$$\alpha_{[r} ... \gamma_{u]} = \frac{1}{p!} \sum_{\Pi} \quad sign(\Pi) \cdot \alpha_{\Pi(r)} ... \gamma_{\Pi(u)} \tag{3}$$

In eq. (3), the sum runs over all permutations Π of the *p* indices *r, … u*, and *sign*(Π) is the sign of the permutation Π, i.e. the factor $(-1)^{\text{number of transpositions of } \Pi}$. Note that the whole terms in eq. (1) and (2) have the same value, but that the coefficients appearing in the sums with the "$dx^r \ldots dx^u$" are (usually) different.

Now to the actual task "Justify that $\varphi \wedge \chi = \alpha \wedge ... \wedge \gamma \wedge \lambda \wedge ... \wedge \nu$ where $\varphi = \alpha \wedge ... \wedge \gamma$ and $\chi = \lambda \wedge ... \wedge \nu$":

According to Penrose, the wedge product of the p-form φ and the q-form χ is defined with antisymmetrized coefficients according to:

$$\varphi \wedge \chi = \sum_{(r,...u,j...m)} \varphi_{[r...u} \chi_{j...m]} \, dx^r \wedge ... \wedge dx^m \tag{4}$$

According to the schema of eq. (2), we have $\varphi_{r...u} = \alpha_{[r} ... \gamma_{u]}$ and $\chi_{j...m} = \lambda_{[j} ... \nu_{m]}$. Hence we can rewrite eq. (4) as:

$$\varphi \wedge \chi = \sum_{(r,...u,j...m)} \alpha_{[[r} ... \gamma_{u]} \lambda_{[j} ... \nu_{m]]} \, dx^r \wedge ... \wedge dx^m \tag{5}$$

On the other hand, the immediate product of all involved 1-forms is, according to the schema of eq. (2), defined as:

$$\alpha \wedge ...\gamma \wedge \lambda \wedge ...\nu = \sum_{(r,...u,j...m)} \alpha_{[r} ... \gamma_u \lambda_j ... \nu_{m]} \, dx^r \wedge ... \wedge dx^m \qquad (6)$$

The assertion "$\varphi \wedge \chi = \alpha \wedge ... \wedge \gamma \wedge \lambda \wedge ... \wedge \nu$ " is hence justified if the coefficients in equations (5) and (6) are equal, i.e. if

$$\alpha_{[r} ... \gamma_u \lambda_j ... \nu_{m]} = \alpha_{[[r} ... \gamma_{u]} \lambda_{[j} ... \nu_{m]]} \qquad (7)$$

Eq. (7) is plausible from the fact that it cannot make a difference

- if the "average" of all terms $\alpha_r ... \gamma_u \lambda_j ... \nu_m$ that result from permutations of the indices is calculated (left side of eq.(7)),
- or if first the "average" of the terms $\alpha_r ... \gamma_u$ and the "average" of the terms $\lambda_j ... \nu_m$ is calculated, and finally the "average" of these averages is calculated (right side of eq.(7)).

More formally, one can show that for any entities $W_{r...uj...m}$ we have:

$$W_{[[r...u]j...m]} = \sum_{\Pi:\ permutation\ in\ M^{p+q}} \frac{1}{(p+q)!} sign(\Pi) \cdot W_{\Pi([r...u]j...m)}$$

$$= \sum_{\Pi:\ permutation\ in\ M^{p+q}} \frac{1}{(p+q)!} sign(\Pi) \sum_{\Gamma:\ permutation\ in\ M^{p}} \frac{1}{p!} sign(\Gamma) \cdot W_{\Pi\circ\Gamma(r...uj...m)}$$

$$= \frac{1}{(p+q)!} \frac{1}{p!} \sum_{\Gamma:\ permutation\ in\ M^{p}} \left(\sum_{\Pi:\ permutation\ in\ M^{p+q}} sign(\Pi) sign(\Gamma) \cdot W_{\Pi\circ\Gamma(r...uj...m)} \right)$$

$$= \frac{1}{(p+q)!} \frac{1}{p!} \sum_{\Gamma:\ permutation\ in\ M^{p}} \left(\sum_{\Pi\circ\Gamma^{-1}:\ permutation\ in\ M^{p+q}} sign(\Pi\circ\Gamma^{-1}) sign(\Gamma) \cdot W_{(\Pi\circ\Gamma^{-1})\circ\Gamma(r...uj...m)} \right)$$

[Note: If Π runs through all permutations in M^{p+q}, then $\Pi\circ\Gamma^{-1}$, does so, too.]

$$= \frac{1}{(p+q)!} \frac{1}{p!} \sum_{\Gamma:\ permutation\ in\ M^{p}} \underbrace{\left(\sum_{\Pi\circ\Gamma^{-1}:\ permutation\ in\ M^{p+q}} sign(\Pi) \underbrace{sign(\Gamma^{-1}) sign(\Gamma)}_{=1} \cdot W_{\Pi(r...uj...m)} \right)}_{=\ p!\ \text{terms independent of}\ \Gamma}$$

$$= \frac{1}{(p+q)!} \left(\sum_{\Pi:\ permutation\ in\ M^{p+q}} sign(\Pi) \cdot W_{\Pi(r...uj...m)} \right) = W_{[r...uj...m]}$$

The first and the last line of the above considerations show that, in general, "inner antisymmetrizations" can be left out (or inserted) at will. This proves eq.(7).