Exercise [16.8]

Statement of the exercise
Show that

$$N = f(a,b) = \frac{1}{2}((a+b)^2 + 3\mathrm{a} + b) \qquad (1)$$

is a one-to-one (bijective) mapping $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$.

Solution
The approach is to explicitly find the inverse functions *a=α(N)* and *b=β(N)*. Then we will show that composing the direct with inverse function and vice-versa we obtain an identity, i.e.

$$f(\alpha(N), \beta(N)) = N$$

$$\alpha(f(a,b)) = a \quad \beta(f(a,b), a) = b$$

This ensures that both direct and inverse functions are bijective (i.e. both injective and surjective), i.e. *f* is a one-to-one mapping.

Lemma
If $s, a \in \mathbb{N}; s \geq a$ the function

$$k(s,a) = \left\lfloor \sqrt{s(s+1) + 2\mathrm{a} + \frac{1}{4}} - \frac{1}{2} \right\rfloor$$

has the property:

$$k(s,a) = s$$

The "floor parenthesis" gives the integer part of the term inside it, in other words it is by definition:

$$k \leq \sqrt{s(s+1) + 2\mathrm{a} + \frac{1}{4}} - \frac{1}{2} < k+1$$

and $k \in \mathbb{N}$. It is then:

$$k(k+1) \leq s(s+1) + 2\mathrm{a} < (k+1)(k+2)$$

Being being $a \in \mathbb{N}$ the right hand inequality implies

$$(k+1)(k+2) > s(s+1) + 2a \geq s(s+1) \quad \Rightarrow \quad k+1 > s \qquad (I)$$

and being *a≤s* the left hand inequality implies :

$$k(k+1) \leq s(s+1) + 2a \leq s(s+1) + 2s \quad \Rightarrow \quad (k-s)(k+1+s) \leq 2s \quad \Rightarrow$$

$$k - s \leq \frac{2s}{k+1+s} < 1 \qquad (II)$$

where in last step (I) was used.
Being *k* and *s* both integers, (I) and (II) imply *k=s*.

Inverse function
Now let's define the functions:

$$\sigma(N)=\left\lfloor \sqrt{2N+\frac{1}{4}}-\frac{1}{2} \right\rfloor$$
$$\alpha(N)=N-\sigma(N)(\sigma(N)+1)/2$$
$$\beta(N)=\sigma(N)-\alpha(N)$$

Bijectivity
Let introduce an auxiliary function:

$$\bar{f}(s,a)=s(s+1)/2+a$$
$$f(a,b)=\bar{f}(a+b,a)$$

Being $a+b\geq a$, we can apply the lemma above as follows:

$$\sigma(f(a,b))=\sigma(\bar{f}(a+b,a))=k(a+b,a)=a+b$$

and therefore is:

$$\alpha(f(a,b))=f-\sigma(f)(\sigma(f)+1)/2=f(a,b)-(a+b)(a+b+1)/2=a$$
$$\beta(f(a,b))=\sigma(f)-\alpha(f)=(a+b)-a=b$$

Moreover

$$f(\alpha(N),\beta(N))=\bar{f}(\alpha(N)+\beta(N),\alpha(N))=\bar{f}(\sigma(N),\alpha(N))=$$
$$=\sigma(\sigma+1)/2+\alpha=\sigma(\sigma+1)/2+N-\sigma(\sigma+1)/2=N$$