

Exercise [16.13]

To reformulate the proof that is given in Penrose's book in Chapter 16.4 (the paragraph beginning "Now for Cantor's astonishing proof ..."), one can express any subset $S(a)$ of the set A (here: the natural numbers) as a sequence of binary digits $[d_1(a) d_2(a) d_3(a) \dots\dots\dots]$, wherein

$$d_k(a) = \begin{cases} 1 & \text{if } k \in S(a) \\ 0 & \text{if } k \notin S(a) \end{cases} \quad (1)$$

All these sequences can be listed in a table with the rows being labelled by a and the columns labelled by k , i.e.:

	k = 1	2	3	4	...	k	...
a = 1	0	0	1	0	...	1	...
2	1	1	1	0	...	0	...
3	1	1	0	1	...	1	...
4	0	0	0	1	...	1	...
...
a	$d_1(a)$	$d_2(a)$	$d_3(a)$	$d_4(a)$...	$d_k(a)$...
...

A subset $S(a)$ that does not contain a is characterized in that it has a "0" on its diagonal cell (yellow cells above). The set

$$Q \text{ of all elements } a \text{ for which } S(a) \text{ does not contain } a \quad (2)$$

can therefore be defined by the binary sequence $[r_1 r_2 r_3 \dots\dots\dots]$ comprising the inverses of all diagonal elements:

$$r_k = \begin{cases} 1 & \text{if } d_k(k) = 0 \\ 0 & \text{if } d_k(k) = 1 \end{cases} \quad (3)$$

The argument can now be brought to an end in two different ways:

A) **DIAGONAL ARGUMENT:**

As Q is a subset of A , it should appear in the table. On the other hand, this cannot be as it differs from every row of the table due its construction (inverting the diagonal element of each row).

→ CONTRADICTION

B) SET ARGUMENT:

As Q is a subset of A , it should appear in the table in some row, say in row q , i.e. we have $Q = S(q)$ and $r_k = d_k(q)$.

Now, the diagonal element $d_q(q)$ cannot be 1, because this would mean that Q contains q due to eq.(1) (which is not possible as, according to (2), Q shall only comprise elements a for which $S(a)$ does not contain a).

However, the diagonal element $d_q(q)$ cannot be 0, either, because this would mean that Q does not contain q due to eq.(1), hence $S(q)$ is a candidate looked for by Q (\rightarrow (2)), hence $q \in Q$.

\rightarrow CONTRADICTION

Remark:

As far as I can see, it is not forbidden that the rows and columns of the considered table are labelled with objects of a set A other than the natural numbers, i.e. having another (higher) cardinality \aleph !?