Exercise [16.8]

Statement of the exercise
Show that

$$N=\frac{1}{2}((a+b)^2+3\mathrm{a}+b) \qquad (1)$$

is a one-to-one mapping $\mathbb{N}\times\mathbb{N}\rightarrow\mathbb{N}$.

Solution
The problem is to show that (1) can be inverted, finding *a* and *b*. Calling *s*=*a*+*b*, (1) can be rewritten as:

$$N=\frac{1}{2}(s(s+1)+2\mathrm{a}) \qquad (1')$$

Let be

$$k=\left\lfloor\sqrt{2N+\frac{1}{4}}-\frac{1}{2}\right\rfloor$$

the integer part of the term within parenthesis. In other words, it is:

$$k\leq\sqrt{2N+\frac{1}{4}}-\frac{1}{2}<k+1$$

i.e.

$$k(k+1)\leq 2N<(k+1)(k+2)$$

Taking into account that *a≥0,* is :

$$(k+1)(k+2)>2N=s(s+1)+2a\geq s(s+1) \quad => \quad k+1>s \qquad (I)$$

and being *a≤s* is :

$$k(k+1)\leq 2N=s(s+1)+2a<s(s+1)+2s \quad => \quad (k-s)(k+1+s)\leq 2s \quad =>$$

$$k-s\leq\frac{2s}{k+1+s}<1 \qquad (II)$$

where in last step (I) was used.
Being *k* and *s* both integers, (I) and (II) imply *k*=*s*, therefore we can finally write:

$$s=\left\lfloor\sqrt{2N+\frac{1}{4}}-\frac{1}{2}\right\rfloor$$

$$a=N-s(s+1)/2$$

$$b=s-a$$