Roger Penrose: The Road To Reality
Chapter 3: Kinds of number in the physical world

Exercise 3.1: Recursion formula for continued fractions

Let $a_1, a_2, \ldots$ be natural numbers and define recursively

$$[a_1] := a_1, \quad [a_1, \ldots, a_n] := a_1 + \frac{1}{[a_2, \ldots, a_n]} \quad (n \geq 2). \tag{1}$$

The **continued fraction** corresponding to the (potentially infinite) series (a_n) may then be formally defined as

$$[a_1, a_2, \ldots] := \lim_{n \to \infty} [a_1, \ldots, a_n]. \tag{2}$$

The expressions $[a_1, \ldots, a_n]$ are called the n-th order approximations of this continued fraction.

(a) Define $p_n, q_n \in \mathbb{N}$ by the recursion relations

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \quad (n \geq 2), \qquad p_1 = a_1, \qquad p_0 = 1, \\ q_n &= a_n q_{n-1} + q_{n-2} \quad (n \geq 2), \qquad q_1 = 1, \qquad q_0 = 0. \end{aligned} \tag{3}$$

Use complete induction to show that for these p_n, q_n the identity

$$\frac{p_n}{q_n} = [a_1, \ldots, a_n]. \tag{4}$$

holds for $n \geq 1$.

Hint: Start from the recursion relation (1) for $[a_1, \ldots, a_{n+1}]$, introduce intermediary quantities $\bar{p}_n$, $\bar{q}_n$ with $\bar{p}_n / \bar{q}_n := [a_2, \ldots, a_{n+1}]$, use the induction assumption on $[a_2, \ldots, a_{n+1}]$ and relate the $\bar{p}_i$ $\bar{q}_i$ to the p_i, q_i to demonstrate that indeed $p_{n+1}/q_{n+1} = [a_1, \ldots, a_{n+1}]$.

(b) Compute the first eight approximations for each of the following continued fractions and convince yourself of the numerical convergence against the 'exact' values given on the left hand side (with $\pi = 3,141592653589793...$):

(i) $\sqrt{2} = [1, 2, 2, 2, 2, \ldots] = [1, \bar{2}]$,

(ii) $7 - \sqrt{3} = [5, 3, 1, 2, 1, 2, \ldots] = [5, 3, \overline{1, 2}]$,

(iii) $\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, \ldots]$.

Remark: the identites (i) and (ii) are proved in Ex. [3.2].

Solution:

(a) For $n = 1$ we have $p_1/q_1 = a_1/1 = a_1 = [a_1]$, so the assertion is obviously true. Similarly, for $n = 2$ we can write

$$\frac{p_2}{q_2} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{a_2 a_1 + 1}{a_2} = a_1 + \frac{1}{a_2} = [a_1, a_2]. \tag{5}$$

Now assume that the identity $p_n/q_n = [a_1, \ldots, a_n]$ – with p_n, q_n defined as in (3) – holds for a certain $n \geq 2$. If one can show that in this case also $p_{n+1}/q_{n+1} = [a_1, \ldots, a_{n+1}]$ holds, then the validity of our assertion will have been proved by induction.

Let's have a closer look at the n-th order approximation $[a_1, \ldots, a_{n+1}]$. Applying the recursion relation (1) yields

$$[a_1, \ldots, a_{n+1}] = a_1 + \frac{1}{[a_2, \ldots, a_{n+1}]}. \tag{6}$$

Here, the finite continued fraction $[a_2, \ldots, a_{n+1}]$ on the right hand side is of order n (not $n+1$), so according to the induction assumption one may write

$$[a_2, \ldots, a_{n+1}] = \frac{\bar{p}_n}{\bar{q}_n}, \tag{7}$$

where the numbers $\bar{p}_i, \bar{q}_i \in \mathbb{N}$ obey the recursion relation

$$\begin{aligned} \bar{p}_i &= a_{i+1}\bar{p}_{i-1} + \bar{p}_{i-2} \quad (i \geq 2), \qquad \bar{p}_1 = a_2, \qquad \bar{p}_0 = 1, \\ \bar{q}_i &= a_{i+1}\bar{q}_{i-1} + \bar{q}_{i-2} \quad (i \geq 2), \qquad \bar{q}_1 = 1, \qquad \bar{q}_0 = 0. \end{aligned} \tag{8}$$

Now let us define a *new* set of numbers p'_i, q'_i, with $0 \leq i \leq n+1$. This definition is again done recursively, namely

$$\begin{aligned} p'_i &:= a_1\bar{p}_{i-1} + \bar{q}_{i-1} \quad (i \geq 1), \qquad p'_0 = 1, \\ q'_i &:= \bar{p}_{i-1} \quad (i \geq 1), \qquad q'_0 = 0. \end{aligned} \tag{9}$$

Note that with these numbers, one may write

$$\frac{p'_{n+1}}{q'_{n+1}} = a_1 + \frac{\bar{q}_n}{\bar{p}_n} = a_1 + \frac{1}{[a_2, \ldots, a_{n+1}]} = [a_1, \ldots, a_{n+1}], \tag{10}$$

i.e. they give us the desired expression for the $(n+1)$-th order approximation of the continued fraction $[a_1, a_2, \ldots]$. Furthermore, they are in fact *identical* to the numbers p_i, q_i.

This last observation already completes our proof, but since it is not fully obvious, we are of course obliged to show it explicitly. So in order to prove the identities $p'_i = p_i$, $q'_i = q_i$, we first reverse (9) to obtain

$$\begin{aligned} \bar{p}_i &= q'_{i+1}, \\ \bar{q}_i &= p'_{i+1} - a_1 q'_{i+1} \end{aligned} \tag{11}$$

for $0 \leq i \leq n$. Now insert this into (8). From the first equation ($\bar{p}_i = a_{i+1}\bar{p}_{i-1} + \bar{p}_{i-2}$) one immediately gets

$$q'_{i+1} = a_{i+1}q'_i + q'_{i-1}, \tag{12}$$

for $2 \leq i \leq n$, whereas from the second ($\bar{q}_i = a_{i+1}\bar{q}_{i-1} + \bar{q}_{i-2}$) one can infer, again for $2 \leq i \leq n$,

$$\begin{aligned} p_{i+1} &= a_1 q'_{i+1} + a_{i+1}(p'_i - a_1 q'_i) + (p'_{i-1} - a_1 q'_{i-1}) \\ &= a_1(a_{i+1}q'_i + q'_{i-1}) + a_{i+1}p'_i - a_1 a_{i+1} q'_i + p'_{i-1} - a_1 q'_{i-1} \\ &= a_{i+1}p'_i + p'_{i-1}, \end{aligned} \tag{13}$$

where in the second step the identity (12) has been used. To summarize, we have shown so far the recursion relations

$$\begin{aligned} p'_i &:= a_i p'_{i-1} + p'_{i-2}, \\ q'_i &:= a_i q'_{i-1} + q'_{i-2} \end{aligned} \tag{14}$$

for $3 \leq i \leq n+1$, which are the same as the recursion relations (3) for the p_i, q_i. Furthermore, for $i = 0, 1, 2$ the identities $p'_i = p_i$, $q'_i = q_i$ can be shown trivially from (9) and (8), so we have indeed etablished $p'_i = p_i$ and $q'_i = q_i$ for all $0 \leq i \leq n+1$.

(b) The following table shows the 'exact' values as well as the approximative values up to the desired order for the continued fractions in question:

n	$\sqrt{2} \simeq 1,414214$		$7 - \sqrt{3} \simeq 5,267949$		$\pi \simeq 3,1415926535897\ldots$	
1	1	1	5	5	3	3
2	$1 + \frac{1}{2}$	1,5	$5 + \frac{1}{3}$	5,33	$3 + \frac{1}{7}$	3,1428
3	$1 + \frac{2}{5}$	1,4	$5 + \frac{1}{4}$	5,25	$3 + \frac{15}{106}$	3,141509
4	$1 + \frac{5}{12}$	1,4167	$5 + \frac{3}{11}$	5,2727	$3 + \frac{16}{113}$	3,141593
5	$1 + \frac{12}{29}$	1,4138	$5 + \frac{4}{15}$	5,2667	$3 + \frac{4687}{33102}$	3,14159265301
6	$1 + \frac{29}{70}$	1,414286	$5 + \frac{11}{41}$	5,2682	$3 + \frac{4703}{33215}$	3,14159265392
7	$1 + \frac{70}{169}$	1,414201	$5 + \frac{15}{56}$	5,267857	$3 + \frac{9390}{66317}$	3,14159265347
8	$1 + \frac{169}{408}$	1,414216	$5 + \frac{41}{153}$	5,267974	$3 + \frac{14093}{99532}$	3,14159265362