Exercise 13.24

Let $B(t)$ be a matrix-valued function of t. We have

$$\det B(t) = \sum_{J\in\pi(n)} \operatorname{sgn}(J) B_{1j_1}(t)\cdots B_{nj_n}(t),$$

where the sum is over all permutations $J = (j_1, \ldots, j_n)$ of the set $\{1, \ldots, n\}$. Thus

$$\begin{aligned}\frac{d}{dt}\det B(t) &= \sum_{J\in\pi(n)} \operatorname{sgn}(J) B'_{1j_1}(t)\cdots B_{nj_n}(t) \\ &\qquad + \cdots + \sum_{J\in\pi(n)} \operatorname{sgn}(J) B_{1j_1}(t)\cdots B'_{nj_n}(t) \\ &= \sum_{i=1}^{n} \det B^{(i)}(t).\end{aligned}$$

Where

$$B^{(i)}_{jk}(t) = \begin{cases} B_{jk}(t) & i \neq j \\ B'_{jk}(t) & i = j\end{cases}$$

(in other words, the i-th row of B is replaced by its derivative).

Now

$$\frac{d}{dt}\det(I + tA)|_{t=0} = \sum_{i=1}^{n} \det(I + tA)^{(i)}|_{t=0}.$$

But

$$(I + tA)^{(i)}_{jk}|_{t=0} = \begin{cases} \delta_{jk}(t) & i \neq j \\ A_{jk}(t) & i = j\end{cases}$$

Hence

$$\det(I + tA)^{(i)}|_{t=0} = A_{ii}$$

and thus

$$\frac{d}{dt}\det(I + tA)|_{t=0} = \operatorname{trace}(A).$$

and of course we also have:

$$\det(I + tA)|_{t=0} = \det I = 1$$

Hence we can write the Taylor expansion:

$$\det(I + \varepsilon A) = 1 + \varepsilon \operatorname{trace}(A) + O(\varepsilon^2).$$