Roger Penrose: The Road To Reality
Chapter 2: An ancient theorem and a modern question

Exercise 2.5: Hemispheric representation of hyperbolic geometry

The **hemispheric representation** of hyperbolic geometry is defined as the inverse image of the conformal representation under the stereographic projection.
Use the known properties of the stereographic projection (see Exercise [2.6]) to show that the hemispheric representation is conformal, with hyperbolic 'straight lines' as vertical semi-circles.

Solution:

Since the stereographic projection is conformal (i.e. preserves angles), it is clear that the angle between two 'straight lines' in the hemispheric representation is always equal to the angle between the corresponding 'straight lines' in the conformal representation. Thus the hemispheric representation is itself conformal.
Furthermore, in the conformal representation hyperbolic 'straight lines' are given by either (i) circle segments intersecting the bounding circle orthogonally, or (ii) straight lines through the origin. In case (ii) the inverse image of such a straight line under the stereographic projection is obviously a vertical semicircle, since all projection rays lie in the same vertical plane. A little less obvious is case (i). From Exercise [2.6] we know that the stereographic projection sends a circle κ' in the equatorial plane to a circle κ on the sphere. But are these circles vertical, i.e. do they intersect the equatorial plane orthogonally? Indeed this must be so, since the stereographic representation is conformal: The 'straight lines' of the conformal representation intersect the bounding circle orthogonally, so there inverse images under the stereographic projection must do the same. But *any* line that is tangential to the sphere at a point on the bounding circle and is also orthogonal to the bounding circle must clearly be a vertical line. So κ is vertical.