Exercise [14.2]

Statement of the exercise

Generalise the axioms

(i) for each a,b,c exists (unique) d such that $[a,b;c,d]$

(ii-a) $[a,b;c,d] \Rightarrow [b,a;d,c]$

(ii-b) $[a,b;c,d] \Rightarrow [a,c;b,d]$

(iii) $[a,b;c,d]$, $[a,b;e,f] \Rightarrow [c,d;e,f]$

for the non Abelian case, i.e. allowing to build a vector space with the first three properties of Exercise 14-1:

(I) $a+(b+c)=(a+b)+c$

(II) it exists a 0 so that, for each a, $0+a+0=a$

(III) for each a it exists a $(-a)$ such that $a+(-a)=0$

but not commutativity:

(IV) $a+b=b+a$

Solution

First a new set of axioms will be derived, showing that they are compatible with the non Abelian case, i.e. that can be demonstrated using (I), (II) and (III) but not (IV).
Then properties (I) (II) and (III) will be demonstrated from this new set of axioms.

Part 1 : finding new axioms.

According to solution of Exercise [14.1], the vector sum $c=a+b$ is defined, choosing an origin o, as:

$[o,a;b,c]$

This tell us that the vector interpretation of $[a;b;c,d]$ is

$(b-a) +(c-a) = (d-a)$ (1)

i.e. it is a parallelogram having one vertex at point a, $(b-a)$ and $(c-a)$ as two sides and d as last vertex; and (1) is just the sum of the vectors lying along the sides, according to parallelogram rule.
We can simplify (1), using only properties (I), (II), (III) and being careful not to exchange order of terms in a sum, as:

$(b-a) +(c-a) = (d-a) \Rightarrow (b-a) +(c-a)+(a-c) = (d-a)+(a-c) \Rightarrow$

$b-a = d-c$ (1a)

This expresses the fact that opposite sides of a parallelogram, interpreted as vectors, are equal.

Using (1a) we can now express the axioms (except the first that just states that in a parallelogram three vertices uniquely define the fourth one) in terms of vector operations:

(iia) $b-a = d-c \Rightarrow a-b = c-d$

(iib) $b-a = d-c \Rightarrow c-a = d-b$

(iii) $b-a = d-c,\ b-a = f-e \Rightarrow d-c= f-e$

Now we can check which of those axioms are compatible with (i.e. that can be demonstrated from) (I), (II) and (III) but not with (IV).
Axiom (ii-a) is compatible with non the Abelian case:

$b\text{-}a = d\text{-}c \Rightarrow (a\text{-}b) + (b\text{-}a) + (c\text{-}d) = (a\text{-}b) + (d\text{-}c) + (c\text{-}d) \Rightarrow c\text{-}d \Rightarrow a\text{-}b$

and also (iii), that simply states transitivity of equality. Therefore they are still valid in the non Abelian case.

Axiom (ii-b) is no more valid in Abelian case, because (see solution of Exercise 14-1) property (IV) follows immediately from it.

Therefore we need to substitute (ii-b) with other axioms. allowing us to demonstrate properties (I), (II) and (III).

An analysis of proof of property (I) in Exercise 14.1 shows that it is possible to substitute, in this proof, axiom (ii-b) with an axiom (iii-b) as follows:

(iii-b) $[a,b;c,d], [a,e;c,f] \Rightarrow [b,e;d,f]$

This axiom is compatible with non Abelian case: subtracting $[a,e;c,f]$ from $[a,b;c,d]$, written in vector terms according to (1a):

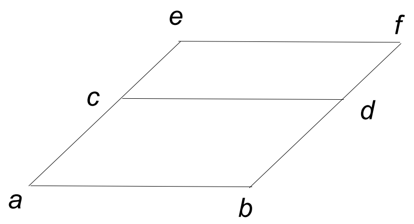
$b\text{-}a = d\text{-}c,\ e\text{-}a = f\text{-}c \Rightarrow (b\text{-}a)\text{-}(e\text{-}a) = (d\text{-}c)\text{-}(f\text{-}c) \Rightarrow b\text{-}a+a\text{-}e = d\text{-}c+c\text{-}f \Rightarrow b\text{-}e = d\text{-}f \Rightarrow e\text{-}b = f\text{-}d$

where the property $-(x+y)= -y-x$ has been used in last two derivations. This property comes out immediately from associativity (III):

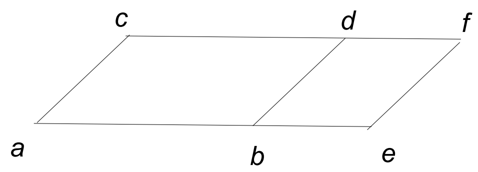
$(x+y)+(-y-x)= x+(y-y)-x=x+o-x=x-x=o$

and therefore $-(x+y)= -y-x$.

A geometrical interpretation of (iii-a) and (iii-b) is given in the following figure.



Axiom iii-a



Axiom iii-b

The axiom (iii-b) was not needed in Abelian case, because it can be demonstrated from (ii-b) and (iii). Indeed, using (ii-b) is:

$[a,b;c,d] \Rightarrow [a,c;b,d]$

$[a,e;c,f] \Rightarrow [a,c;e,f]$

and these two, using (iii), imply $[b,d;e,f]$ that is, again using (ii-b), $[b,e;d,f]$

So removing (ii-b) and adding (iii-b) lead to a weaker set of axioms.

An analysis of proof of property (II) in Exercise 14.1 shows that it is possible to substitute, in this proof, axiom (ii-b) with an axiom (ii-c) as follows:

(ii-c) $[a,b;c,d] \Rightarrow [c,d;a,b]$

This axiom is compatible with Abelian case, being simply the symmetric property of equality:

$b\text{-}a = d\text{-}c \Rightarrow d\text{-}c = b\text{-}a$

Note that in Abelian case this axiom is not needed, because it can be demonstrated from (ii-a) and (ii-b): $[a,b;c,d]$ => $[a,c;b,d]$ => $[c,a;d,b]$ => $[c,d;a,b]$
So removing (ii-b) and adding (ii-c) lead to a weaker set of axioms.

Demonstration of (III) was not directly depending by (ii-b).

Therefore a new set of axioms for non Abelian case is:

(i) for each a,b,c exists (unique) d such that $[a,b;c,d]$

(ii-a) $[a,b;c,d]$ => $[b,a;d,c]$

(ii-c) $[a,b;c,d]$ => $[c,d;a,b]$

(iii-a) $[a,b;c,d]$, $[a,b;e,f]$ => $[c,d;e,f]$

(iii-b) $[a,b;c,d]$, $[a,e;c,f]$ => $[b,e;d,f]$

As we have seen above, these new axioms are compatible with non Abelian case and are weaker than the original set of axioms.

Part 2: define vector space and demonstrate its properties (I) (II) and (III) with the new axioms.

2-A-Definition of sum (same as in Exercise [14.1])
Fixed a point o as origin, we can define

$c=a+b$

where c the point, that always exists and is unique because of (i), closing the parallelogram $[o,a;b,c]$.

2.B-Demonstration of (I)
Let be

$b+c=d$	i.e. $[o,b;c,d]$	(1)
$a+d=f$	i.e. $[o,a;d,f]$	(2)
$a+b=e$	i.e. $[o,a;b,e]$	(3)
$e+c=g$	i.e. $[o,e;c,g]$	(4)

We have to demonstrate that $f=g$.
From (3) and (2) using (iii-a) is $[b,e;d,f]$ and from (1) and (4) using (iii-b) is $[b,e;d,g]$; these two and (i) demonstrate that $f=g$.

2.C- Demonstration of (II)
For any a and b , because of axiom (i) there is a c such that $[o,a;b,c]$. Applying (ii-c) this implies $[b,c;o,a]$; applying (iii-a) to this parallelogram: $[b,c;o,a]$, $[b,c;o,a]$ => $[o,a;o,a]$ i.e. for any a is:

$a+o=a$.

2.D-Demonstration of (III) (same as in Exercise [14.1])
For any a, because of axiom (i) , there is b such that $[a,o;o,b]$. From (ii-a) is then $[o,a;b,o]$ i.e.

$a+b=o$

and therefore b is the opposite ($-a$) of a.