Exercise [14.1]

Statement of the exercise

Given the axioms:

(i) for each a,b,c exists (unique) d such that $[a,b;c,d]$

(ii-a) $[a,b;c,d] \Rightarrow [b,a;d,c]$

(ii-b) $[a,b;c,d] \Rightarrow [a,c;b,d]$

(iii) $[a,b;c,d]$, $[a,b;e,f] \Rightarrow [c,d;e,f]$

demonstrate that, fixing an origin o, it is possible to define a "vector sum" $(a+b)$ with the following properties:

(I) $a+(b+c)=(a+b)+c$

(II) it exists a vector 0 so that, for each a, $a+0=a$

(III) for each a there is an opposite $(-a)$ such that $a+(-a)=0$

(IV) $a+b=b+a$

Note that (II) and (III) are enough, because together with (I) they imply also $0+a=a$ and $(-a)+a=0$ as per exercise [13.1].

Solution

A-Definition of sum

Fixed a point o as origin, we can define

$c=a+b$

where c the point, that always exists and is unique because of (i), closing the parallelogram $[o,a;b,c]$.

B-Demonstration of (I)

Let be

$b+c=d$	i.e. $[o,b;c,d]$	(1)
$a+d=f$	i.e. $[o,a;d,f]$	(2)
$a+b=e$	i.e. $[o,a;b,e]$	(3)
$e+c=g$	i.e. $[o,e;c,g]$	(4)

We have to demonstrate that $f=g$.

From (3) and (2) using (iii) is $[b,e;d,f]$.

From (1) and (ii-b) is $[o,c;b,d]$; and from (4) and (ii-b) is $[o,c;e,g]$; then using (iii) is $[b,d;e,g]$ and using (ii-b) is $[b,e;d,g]$.

Being the fourth vertex in (i) unique, from $[b,e;d,f]$ and $[b,e;d,g]$ follows $f=g$.

C- Demonstration of (II)

For any a and b , because of axiom (i) there is always a c such that $[o,a;b;c]$. Applying (ii-a) and (ii-b):

$[o,a;b;c] \Rightarrow [o,b;a,c] \Rightarrow [b,o;c,a] \Rightarrow [b,c;o,a]$.

Applying (iii) to parallelogram $[b,c;o,a]$ is:

$[b,c;o,a]$, $[b,c;o,a] \Rightarrow [o,a;o,a] \Rightarrow a+o=a$

i.e. the zero vector is o.

D-Demonstration of (III)
For any a, because of (i), there is b such that $[a,o;\ o,b]$. From (ii-a) is $[o,a;\ b,\ o\ ]$, i.e.

$$a+b=o$$

and therefore b is the opposite ($-a$) of a.

E- Demonstration of (IV)
Using (ii-b) is:

$$a+b=c \iff [o,a;b,c] \Rightarrow [o,b;a,c] \iff b+a=c$$

therefore

$$a+b=b+a$$