Exercise 14.32

This proof relies on the following result which I will prove afterwards:

$$\pounds_{[X,Y]} = \pounds_X \pounds_Y - \pounds_Y \pounds_X \qquad (*)$$

Assuming this result for the moment, let κ and η be Killing vectors and thus, $\pounds_\kappa g = 0 = \pounds_\eta g$. We then have:

$$\pounds_{[\kappa,\eta]} g = \pounds_\kappa(\pounds_\eta g) - \pounds_\eta(\pounds_\kappa g) = \pounds_\kappa 0 - \pounds_\eta 0 = 0$$

which proves that $[\kappa, \eta]$ is a Killing vector.

It remains to show that $(*)$ holds. First of all notice that for a scalar field Φ we have:

$$\begin{aligned}
\pounds_{[X,Y]}\Phi &= [X,Y]\Phi \\
&= X(Y(\Phi)) - Y(X(\Phi)) \\
&= \pounds_X(\pounds_Y \Phi) - \pounds_Y(\pounds_X \Phi) \\
&= (\pounds_X \pounds_Y - \pounds_Y \pounds_X)\Phi.
\end{aligned}$$

Similarly, for a vector field Z we have:

$$\begin{aligned}
(\pounds_X \pounds_Y - \pounds_Y \pounds_X)Z &= \pounds_X[Y,Z] - \pounds_Y[X,Z] \\
&= [X,[Y,Z]] - [Y,[X,Z]] \\
&= -[[Y,Z],X] - [[Z,X],Y] \\
&= [[X,Y],Z] \\
&= \pounds_{[X,Y]} Z
\end{aligned}$$

To prove for an arbitrary tensor field T, suppose the result is true for the tensor S and the contraction $T \cdot S$. From the definition of the Lie derivative we have:

$$\begin{aligned}
(\pounds_{[X,Y]}T) \cdot S &= \pounds_{[X,Y]}(T \cdot S) - T \cdot \pounds_{[X,Y]} S \\
&= (\pounds_X \pounds_Y - \pounds_Y \pounds_X)(T \cdot S) - T \cdot (\pounds_X \pounds_Y - \pounds_Y \pounds_X)S \\
&= \pounds_X((\pounds_Y T) \cdot S + T \cdot \pounds_Y S) - \pounds_Y((\pounds_X T) \cdot S + T \cdot \pounds_X S) \\
&\quad - T \cdot (\pounds_X \pounds_Y - \pounds_Y \pounds_X)S \\
&= (\pounds_X \pounds_Y T) \cdot S + \pounds_Y T \cdot \pounds_X S + \pounds_X T \pounds_Y S + T \cdot \pounds_Y \pounds_X S \\
&\quad - (\pounds_Y \pounds_X T) \cdot S - \pounds_X T \cdot \pounds_Y S - \pounds_Y T \pounds_X S - T \cdot \pounds_X \pounds_Y S \\
&\quad - T \cdot (\pounds_X \pounds_Y - \pounds_Y \pounds_X)S \\
&= ((\pounds_X \pounds_Y - \pounds_Y \pounds_X)T) \cdot S
\end{aligned}$$

Because we have shown the result for scalar fields and vector fields, this shows that the result $(*)$ can be established via induction on any arbitrary tensor field.