Exercise [14.34]

Statement of the exercise

For symplectic case:

$$S_{ab}=-S_{ba} \qquad (1)$$

$$\nabla_a S_{bc}+\nabla_b S_{ca}+\nabla_c S_{ab}=0 \qquad (2)$$

$$S^{ab} S_{bc}=\delta^a_c \qquad (3)$$

$$\{\Phi,\Psi\}=\frac{1}{2}S^{ab}\nabla_a\Phi\nabla_b\Psi \qquad (4)$$

show that

A) $\{\Phi,\Psi\}=-\{\Psi,\Phi\}$

B) $\{\Theta,\{\Phi,\Psi\}\}+\{\Phi,\{\Psi,\Theta\}\}+\{\Psi,\{\Theta,\Phi\}\}=0$

Solution – Part A

First we need to show that (1) holds also for S^{ab}.

$$S^{ab}=\delta^a_c S^{cb}=(S_{cd}S^{da})S^{cb}=-S_{dc}S^{da}S^{cb}=-\delta^b_d S^{da}=-S^{ba} \qquad (1a)$$

Using this result:

$$\{\Phi,\Psi\}=\frac{1}{2}S^{ab}\nabla_a\Phi\nabla_b\Psi=\frac{1}{2}S^{ba}\nabla_b\Phi\nabla_a\Psi=-\frac{1}{2}S^{ab}\nabla_b\Phi\nabla_a\Psi=-\{\Psi,\Phi\}$$

where first the names of contracted indices were exchanged and then antisymmetry (1a) was used.

Solution – Part B

It is suggested to check first that :

$$S^{a[b}\nabla_a S^{cd]}=0 \qquad (I)$$

We can compute the expression finding first a formula providing derivative of $\mathbf{S}^{-1}$ in terms of derivative of $\mathbf{S}$. For that, we can observe that, because of result of exercise [14.6], in every connection is:

$$\nabla_a(\delta^b_c)=\partial_a\delta^b_c+\Gamma^b_{as}\delta^s_c-\Gamma^s_{ac}\delta^b_s=0+\Gamma^b_{ac}-\Gamma^b_{ac}=0$$

therefore:

$$0=\nabla_a(\delta^b_q)=\nabla_a(S^{bp}S_{pq})=\nabla_a(S^{bp})S_{pq}+S^{bp}\nabla_a S_{pq} \;\Rightarrow\; \nabla_a(S^{bp})S_{pq}=-S^{bp}\nabla_a S_{pq}$$

$$\Rightarrow\; \nabla_a(S^{bp})S_{pq}S^{qc}=-S^{bp}S^{qc}\nabla_a S_{pq} \;\Rightarrow\; \nabla_a(S^{bp})\delta^c_p=S^{bp}S^{cq}\nabla_a S_{pq} \qquad (II)$$

$$\Rightarrow\; \nabla_a S^{bc}=S^{bp}S^{cq}\nabla_a S_{pq}$$

Then we can compute the lefth hand side of (I):

$$S^{a[b}\nabla_a S^{cd]}=S^{ab}\nabla_a S^{cd}+S^{ac}\nabla_a S^{db}+S^{ad}\nabla_a S^{bc}=S^{ab}S^{cp}S^{dq}\nabla_a S_{pq}+S^{ac}S^{dp}S^{bq}\nabla_a S_{pq}+S^{ad}S^{bp}S^{cq}\nabla_a S_{pq}$$

where first the expression was expanded taking into account antisymmetry (1a) and then formula

(II) was used. Cyclically rotating the contracted indices a, p, q is:

$$S^{a[b}\nabla_a S^{cd]} = S^{ab}S^{cp}S^{dq}\nabla_a S_{pq} + S^{pc}S^{dq}S^{ba}\nabla_p S_{qa} + S^{qd}S^{ba}S^{cp}\nabla_q S_{ap} =$$
$$= S^{ab}S^{cp}S^{dq}(\nabla_a S_{pq} + \nabla_p S_{qa} + \nabla_q S_{ap}) = 0$$

where (2) was used.

Is then:

$$\{\Theta, \{\Phi, \Psi\}\} = \frac{1}{2}S^{ab}\nabla_a\Theta\nabla_b(\frac{1}{2}S^{cd}\nabla_c\Phi\nabla_d\Psi) =$$
$$= \frac{1}{4}S^{ab}\nabla_a\Theta(\nabla_b S^{cd}\nabla_c\Phi\nabla_d\Psi + S^{cd}\nabla_c\nabla_b\Phi\nabla_d\Psi + S^{cd}\nabla_c\Phi\nabla_b\nabla_d\Psi) =$$
$$= -\frac{1}{4}S^{ba}\nabla_b S^{cd}\nabla_a\Theta\nabla_c\Phi\nabla_d\Psi + \frac{1}{4}S^{ab}S^{cd}\nabla_a\Theta(\nabla_c\nabla_b\Phi\nabla_d\Psi + \nabla_c\Phi\nabla_b\nabla_d\Psi)$$

When performing the cyclical sum (B) the first term, cyclically rotating the contracted indices *a, c, d* to follow the cyclical rotation of Θ, Φ, Ψ, becomes:

$$-\frac{1}{4}S^{b[a}\nabla_b S^{cd]}\nabla_a\Theta\nabla_c\Phi\nabla_d\Psi$$

and it vanishes because of (I). The remaining terms:

$$\frac{1}{4}S^{ab}S^{cd}\nabla_c\nabla_b\Phi\nabla_a\Theta\nabla_d\Psi + \frac{1}{4}S^{ab}S^{cd}\nabla_b\nabla_d\Psi\nabla_a\Theta\nabla_c\Phi$$

cyclically summed becomes:

$$\frac{1}{4}S^{ab}S^{cd}\nabla_c\nabla_b\Phi\nabla_a\Theta\nabla_d\Psi + \frac{1}{4}S^{ab}S^{cd}\nabla_b\nabla_d\Psi\nabla_a\Theta\nabla_c\Phi +$$
$$\frac{1}{4}S^{ab}S^{cd}\nabla_c\nabla_b\Theta\nabla_a\Psi\nabla_d\Phi + \frac{1}{4}S^{ab}S^{cd}\nabla_b\nabla_d\Phi\nabla_a\Psi\nabla_c\Theta +$$
$$\frac{1}{4}S^{ab}S^{cd}\nabla_c\nabla_b\Psi\nabla_a\Phi\nabla_d\Theta + \frac{1}{4}S^{ab}S^{cd}\nabla_b\nabla_d\Theta\nabla_a\Phi\nabla_c\Psi$$

If we change the names of contracted indices in first term of each line as follows: $a \to c$, $b \to d$, $c \to b$, $d \to a$:

$$\frac{1}{4}S^{cd}S^{ba}\nabla_b\nabla_d\Phi\nabla_c\Theta\nabla_a\Psi + \frac{1}{4}S^{ab}S^{cd}\nabla_b\nabla_d\Psi\nabla_a\Theta\nabla_c\Phi +$$
$$\frac{1}{4}S^{cd}S^{ba}\nabla_b\nabla_d\Theta\nabla_c\Psi\nabla_a\Phi + \frac{1}{4}S^{ab}S^{cd}\nabla_b\nabla_d\Phi\nabla_a\Psi\nabla_c\Theta +$$
$$\frac{1}{4}S^{cd}S^{ba}\nabla_b\nabla_d\Psi\nabla_c\Phi\nabla_a\Theta + \frac{1}{4}S^{ab}S^{cd}\nabla_b\nabla_d\Theta\nabla_a\Phi\nabla_c\Psi$$

then the 1st term of 1st line cancels with 2nd term of 2nd line, the 1st term of 2nd line cancels with 2nd term of 3rd line, the 1st term of 3rd line cancels with 2nd term of 1st line, therefore (B) is verified.