Exercise [14.28]

Statement of the exercise

If I understood correctly, the exercise ask to demonstrate that a metric **g** completely defines geodesic lines, more precisely:

1) Shortest path lines are geodesic

Given a metric **g** , the line between two points having the minimum length

$$l=\int_A^B \sqrt{g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}}\,du \qquad (1)$$

(here expressed in term of a parameter *u* as clarified in note 11) is a geodesic line as defined in chapter 14.5, i.e. satisfies equation:

$$t^a\nabla_a t^b \propto t^b \quad \text{i.e.} \quad t^a\nabla_a t^b = k\,t^b \qquad (2)$$

where k is an arbitrary proportionality factor.

2) Affine parameters are defined by the metric

If parameter u in (1) is chosen to be s, the length of the curve, then it is an affine parameter,

$$t^a\nabla_a t^b=0 \qquad (3)$$

Solution-Part 1

The tangent of a line, given in parametric form $x^a(u)$ is $t^a=\frac{dx^a}{du}$, therefore (1) can be rewritten as $l=\int_A^B\sqrt{g_{ab}t^a t^b}\,du$.

The Euler-Lagrange equation of the variational problem of finding the minimum of (1) is:

$$\frac{d}{du}\left(\frac{\partial}{\partial t^c}\sqrt{g_{ab}t^a t^b}\right)-\frac{\partial}{\partial x^c}\left(\sqrt{g_{ab}t^a t^b}\right)=0 \qquad (1')$$

Renaming $R=\sqrt{g_{ab}t^a t^b}$, that is assumed not vanishing (see note), we have:

$$\frac{d}{du}\left(\frac{\partial R}{\partial t^c}\right)=\frac{d}{du}\left(\frac{g_{ac}t^a+g_{cb}t^b}{2\mathrm{R}}\right)=$$

$$=-\frac{g_{ac}t^a+g_{cb}t^b}{2\mathrm{R}^2}\frac{dR}{du}+\frac{1}{2\mathrm{R}}\left(\frac{\partial g_{ac}}{\partial x^b}\frac{dx^b}{du}t^a+\frac{\partial g_{cb}}{\partial x^a}\frac{dx^a}{du}t^b\right)+\frac{1}{2\mathrm{R}}\left(g_{ac}\frac{dt^a}{du}+g_{cb}\frac{dt^b}{du}\right)=$$

$$=-\frac{g_{cb}}{R^2}t^b\frac{dR}{du}+\frac{1}{2\mathrm{R}}\left(\frac{\partial g_{ac}}{\partial x^b}+\frac{\partial g_{cb}}{\partial x^a}\right)t^a t^b+\frac{g_{cb}}{R}\frac{dt^b}{du}$$

$$\frac{\partial R}{\partial x^c}=\frac{1}{2\mathrm{R}}\frac{\partial g_{ab}}{\partial x^c}t^a t^b$$

and therefore (1') becomes:

$$\frac{1}{2\mathrm{R}}\left(\frac{\partial g_{ac}}{\partial x^b}+\frac{\partial g_{cb}}{\partial x^a}-\frac{\partial g_{ab}}{\partial x^c}\right)t^a t^b+\frac{g_{cb}}{R}\frac{dt^b}{du}=\frac{g_{cb}}{R^2}t^b\frac{dR}{du}$$

and multiplying by (R g^{dc})

$$\frac{1}{2}g^{dc}\left(\frac{\partial g_{ac}}{\partial x^b}+\frac{\partial g_{cb}}{\partial x^a}-\frac{\partial g_{ab}}{\partial x^c}\right)t^a t^b+\frac{dt^d}{du}=\frac{1}{R^2}\frac{dR}{du}t^d$$

Using result of exercise [14.26], the equation of minimum length line is finally:

$$\frac{d\,t^d}{du}+\Gamma^d_{ab}t^a t^b=\frac{1}{R}\frac{dR}{du}t^d \qquad (1'')$$

Using result of exercise [14.6], left hand side of equation (2) becomes:

$$t^a\nabla_a t^b=t^a\left(\frac{\partial t^b}{\partial x^a}+\Gamma^b_{ac}t^c\right)=\frac{d\,t^b}{du}+\Gamma^b_{ac}t^a t^c$$

i.e. equation (2) can be rewritten as:

$$\frac{d\,t^b}{du}+\Gamma^b_{ac}t^a t^c=k\,t^b \qquad (2')$$

Clearly (1") and (2') are the same equation, with the position $k=\frac{1}{R}\frac{dR}{du}=\frac{1}{\sqrt{g_{ab}t^a t^b}}\frac{d}{du}\sqrt{g_{ab}t^a t^b}$.

Solution-Part 2

Parameter is affine when (3) is satisfied, i.e. when $\frac{1}{R}\frac{dR}{du}=0$, that implies the square root being a constant:

$$\sqrt{g_{ab}t^a t^b}=h \qquad (3')$$

Inserting this back in (1) we obtain:

$$l=\int_A^B h\,du=h\,u_B-h\,u_A$$

This means that when there is a metrics the parameter is affine only when it is the length of the curve, apart for a scale factor and a constant term, coherently with result of exercise [14.13].

NOTE: This assume R≠0, i.e. that the measure of distance is not vanishing: this is, I think, the problem mentioned in chapter 14.5 about light rays in relativity.