Exercise [14.5]

Statement of the exercise

The exercise asks to demonstrate that the differences between two different covariant derivatives $\overset{1}{\nabla}_b$ and $\overset{2}{\nabla}_b$ applied to any tensor **T** can always be expressed in terms of **T** and of a quantity Γ^a_{bc} .

Solution

The solution follows the three steps suggested in the book.

A) Find difference between a $\overset{1}{\nabla}_b$ and $\overset{2}{\nabla}_b$ applied to a vector field ξ^a.

By definition, the covariant derivative along a direction **w**, i.e. $w^b \overset{1}{\nabla}_b \xi^a$, in a point P is evaluated this way:

- find the point Q, displaced from P by an infinitesimal amount ε along the direction **w;**
- find the vector ξ(Q), evaluated at point Q;
- find the vector ξ(P), evaluated at point P;
- find the vector π(P,**w**), obtained from ξ(P) by parallel transport from P to Q;
- evaluate the difference ξ(Q)-π(P,**w**) and then take the limit when ε tend to zero.

In symbols, omitting the limit,

$$w^b \overset{1}{\nabla}_b \xi^a = \xi^a(Q) - \overset{1}{\pi}{}^a(\xi, \boldsymbol{w})$$

Vector $\xi^a(Q)$ depends of course only by the vector field, not by the way the covariant derivative is defined; vector $\pi^a(\xi,\mathbf{w})$ instead depends by the definition of the covariant derivative, i.e. by the way we do parallel transport.

Therefore, taking the difference between two covariant derivatives along **w** :

$$w^b(\overset{1}{\nabla}_b - \overset{2}{\nabla}_b)\xi^a = -\overset{1}{\pi}{}^a(\xi, \boldsymbol{w}) + \overset{2}{\pi}{}^a(\xi, \boldsymbol{w})$$

This shows that the difference between two covariant derivatives along **w** is a function only of the direction **w** and the vector ξ(P), evaluated at point P.

The right hand side of the above expression is:

- linear with respect to the direction **w**, with components w^b ;
- linear with respect to the vector ξ, with components ξ^a ;

because these are properties of the covariant derivative along **w**, and therefore also of the difference between two of them.

The left hand side has therefore the same properties , therefore we can write, making explicit the indices:

$$w^b(\overset{1}{\nabla}_b - \overset{2}{\nabla}_b)\xi^a = \Gamma^a_{bc}\xi^c w^b \qquad (1)$$

Note that indices *c* and *b* are tensorial indices, i.e. **Γ** is a bilinear function of w^b and ξ^c, giving as a result a vector. This does not happen for upper index *a*, that only allows to identify the components of the resulting vector; but nothing tell us that Γ^a_{bc} is a $[^1{}_2]$ tensor: in other words, is not proven that it is a multilinear function of two vectors and one covector.

Being (1) true for every vector w^b we can also write

$$(\overset{1}{\nabla}_b - \overset{2}{\nabla}_b)\xi^a = \Gamma^a_{bc}\xi^c \qquad (1')$$

B) Find difference between $\overset{1}{\nabla}_b$ and $\overset{2}{\nabla}_b$ applied to a covector field α_a.
We can use the Leibniz rule on scalar product:

$$(\overset{1}{\nabla}_b-\overset{2}{\nabla}_b)(\alpha_c\xi^c)=((\overset{1}{\nabla}_b-\overset{2}{\nabla}_b)\alpha_c)\xi^c+\alpha_a((\overset{1}{\nabla}_b-\overset{2}{\nabla}_b)\xi^a)$$

where the names of contracted indices were changed in second term of right hand side.
The covariant derivative of a scalar is, by definition, the exterior derivative d, and therefore does not depend by way we do the parallel transport; hence the left hand side is zero. The expression becomes:

$$((\overset{1}{\nabla}_b-\overset{2}{\nabla}_b)\alpha_c)\xi^c=-\alpha_a((\overset{1}{\nabla}_b-\overset{2}{\nabla}_b)\xi^a)=-\alpha_a\Gamma^a_{bc}\xi^c$$

and being this true for every vector ξ it is

$$(\overset{1}{\nabla}_b-\overset{2}{\nabla}_b)\alpha_c=-\alpha_a\Gamma^a_{bc} \tag{2}$$

C) Find difference between $\overset{1}{\nabla}_v$ and $\overset{2}{\nabla}_v$ applied to a generic $[^p{}_q]$ tensor **T**
Tensor **T** is completey defined by its action on q vectors and p covectors, that gives a scalar Φ:

$$\Phi=T^{a..c}_{r..u}\alpha_a..\gamma_c\zeta^{r..}\eta^u$$

Applying the Leibniz rule:

$$\begin{aligned}(\overset{1}{\nabla}_v-\overset{2}{\nabla}_v)(T^{a..c}_{r..u}\alpha_a..\gamma_c\zeta^{r..}\eta^u)=&((\overset{1}{\nabla}_v-\overset{2}{\nabla}_v)T^{a..c}_{r..u})\alpha_a..\gamma_c\zeta^{r..}\eta^u+\\ &T^{a..c}_{r..u}((\overset{1}{\nabla}_v-\overset{2}{\nabla}_v)\alpha_a)..\gamma_c\zeta^{r..}\eta^u+..+T^{a..c}_{r..u}\alpha_a..((\overset{1}{\nabla}_v-\overset{2}{\nabla}_v)\gamma_c)\zeta^{r..}\eta^u+\\ &T^{a..c}_{r..u}\alpha_a..\gamma_c((\overset{1}{\nabla}_v-\overset{2}{\nabla}_v)\zeta^r)..\eta^u+..+T^{a..c}_{r..u}\alpha_a..\gamma_c\zeta^{r..}((\overset{1}{\nabla}_v-\overset{2}{\nabla}_v)\eta^u)\end{aligned}$$

Taking into account that $\overset{1}{\nabla}_v-\overset{2}{\nabla}_v$ applied to a scalar is 0, as in part B, and using (1') and (2) we can rewrite as:

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$$\begin{aligned}&((\overset{1}{\nabla}_v-\overset{2}{\nabla}_v)T^{a..c}_{r..u})\alpha_a..\gamma_c\zeta^{r..}\eta^u=\\ &=T^{a..c}_{r..u}(\Gamma^s_{va}\alpha_s)..\gamma_c\zeta^{r..}\eta^u+..+T^{a..c}_{r..u}\alpha_a..(\Gamma^s_{vc}\gamma_s)\zeta^{r..}\eta^u-T^{a..c}_{r..u}\alpha_a..\gamma_c(\Gamma^r_{vs}\zeta^s)..\eta^u-..-T^{a..c}_{r..u}\alpha_a..\gamma_c\zeta^{r..}(\Gamma^u_{vs}\eta^s)=\\ &=T^{s..c}_{r..u}(\Gamma^a_{vs}\alpha_a)..\gamma_c\zeta^{r..}\eta^u+..+T^{a..s}_{r..u}\alpha_a..(\Gamma^c_{vs}\gamma_c)\zeta^{r..}\eta^u-T^{a..c}_{s..u}\alpha_a..\gamma_c(\Gamma^s_{vr}\zeta^r)..\eta^u-..-T^{a..c}_{r..s}\alpha_a..\gamma_c\zeta^{r..}(\Gamma^s_{vu}\eta^u)\end{aligned}$$

where some contracted index name has been changed in last line.

Being this expression true for every set of vectors and covectors, we obtain the final result:

$$(\overset{1}{\nabla}_v-\overset{2}{\nabla}_v)T^{a..c}_{r..u}=\boldsymbol{T}^{s..c}_{r..u}\Gamma^a_{vs}+..+\boldsymbol{T}^{a..s}_{r..u}\Gamma^c_{vs}-\boldsymbol{T}^{a..c}_{s..u}\Gamma^s_{vr}-..-\boldsymbol{T}^{a..c}_{r..s}\Gamma^s_{vu} \tag{3}$$