Exercise 14.23

Let α be a p-form and ∇ a torsion free connection. We have

$$\nabla_{i_1}\alpha_{i_2\ldots i_{p+1}} = \partial_{i_1}\alpha_{i_2\ldots i_{p+1}} - \alpha_{ji_3\ldots i_{p+1}}\Gamma^{j}_{i_2 i_1} - \cdots - \alpha_{i_2\ldots i_{p+1}j}\Gamma^{j}_{i_{p+1}i_1}$$

where $\Gamma^a_{cb} := [\nabla\partial_c]^a_b$ (see the solution to Exercise 14.4).

Hence

$$\nabla_{[i_1}\alpha_{i_2\ldots i_{p+1}]} = \partial_{[i_1}\alpha_{i_2\ldots i_{p+1}]} - \sum_{(i_1,\ldots,i_{p+1})\in\pi(p+1)} \frac{\mathrm{sgn}(\pi)}{(p+1)!}(\alpha_{ji_3\ldots i_{p+1}}\Gamma^{j}_{i_2 i_1} - \cdots - \alpha_{i_2\ldots i_{p+1}j}\Gamma^{j}_{i_{p+1}i_1})$$

where the sum is over all permutations of $p+1$ symbols and $\mathrm{sgn}(\pi)$ denotes the sign of the permutation.

Now

$$\partial_{[i_1}\alpha_{i_2\ldots i_{p+1}]} = (d\alpha)_{i_1\ldots i_{p+1}}$$

In the sum, for each term of the form:

$$\alpha_{i_1\ldots i_{k-1}ji_{k+1}\ldots i_{p+1}}\Gamma^{j}_{i_k i_1}$$

there also will be a term:

$$-\alpha_{i_1\ldots i_{k-1}ji_{k+1}\ldots i_{p+1}}\Gamma^{j}_{i_1 i_k}$$

where the sign is opposite because the permutations have opposite parity. Since the torsion vanishes (and so $\Gamma^{j}_{i_k i_1} = \Gamma^{j}_{i_1 i_k}$), these terms will cancel.

Thus we have

$$(d\alpha)_{i_1\ldots i_{p+1}} = \nabla_{[i_1}\alpha_{i_2\ldots i_{p+1}]}$$