

$\Phi(z) = \alpha + i\beta$ , where  $\alpha = \alpha(x, y)$  and  $\beta = \beta(x, y)$ .

From the definition of the derivative, we can write

$$\frac{d\Phi}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Phi(z + \Delta z) - \Phi(z)}{\Delta z}, \text{ where } \Delta z = \Delta x + i\Delta y.$$

$$\begin{aligned} \text{Now } \lim_{\Delta z \rightarrow 0} \frac{\Phi(z + \Delta z) - \Phi(z)}{\Delta z} &= \\ \lim_{\Delta z \rightarrow 0} \frac{\alpha(x + \Delta x, y + \Delta y) - \alpha(x, y) + i\{\beta(x + \Delta x, y + \Delta y) - \beta(x, y)\}}{\Delta x + i\Delta y} & \quad (1) \end{aligned}$$

Remember from the definition of derivative that  $\lim_{\Delta z \rightarrow 0}$  must be the same *regardless of the manner in which*  $\Delta z \rightarrow 0$ .

So let  $\Delta z \rightarrow 0$  along the  $x$  direction, so that  $\Delta y = 0$  and  $\Delta z = \Delta x$ .

We can then write (1) as

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{\alpha(x + \Delta x, y) - \alpha(x, y) + i\{\beta(x + \Delta x, y) - \beta(x, y)\}}{\Delta x} & \text{ or as} \\ \lim_{\Delta z \rightarrow 0} \frac{\alpha(x + \Delta x, y) - \alpha(x, y)}{\Delta x} + i \lim_{\Delta z \rightarrow 0} \frac{\beta(x + \Delta x, y) - \beta(x, y)}{\Delta x} & \\ = \frac{\partial \alpha}{\partial x} + i \frac{\partial \beta}{\partial x} & \quad (2) \end{aligned}$$

In a similar manner let  $\Delta z \rightarrow 0$  along the  $y$  direction, so that  $\Delta x = 0$  and  $\Delta z = i\Delta y$ . We can then write (1) as

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{\alpha(x, y + \Delta y) - \alpha(x, y) + i\{\beta(x, y + \Delta y) - \beta(x, y)\}}{i\Delta y} & \text{ or as} \\ \lim_{\Delta z \rightarrow 0} \frac{\alpha(x, y + \Delta y) - \alpha(x, y)}{i\Delta y} + i \lim_{\Delta z \rightarrow 0} \frac{\beta(x, y + \Delta y) - \beta(x, y)}{i\Delta y} & \\ = -i \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial y} & \quad (3) \end{aligned}$$

Since (2) and (3) must be equal, it follows that

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y} \text{ and } \frac{\partial \beta}{\partial x} = -\frac{\partial \alpha}{\partial y}$$

which are the Cauchy-Riemann equations.