Note: Penrose uses two extended complex planes in his preamble to this exercise, the t-plane and the z-plane. We will use this notation in what follows. Please refer also to figure 8.7 in RTR for an explanation of stereographic projection and the Riemann sphere.

What Penrose is asking us to do

Penrose explains, in section 8.3 of RTR, that on the Riemann sphere (obtained by stereographic projection of the extended complex plane), the real axis of the extended complex plane is a circle, specifically the great circle through the north and south poles of the sphere and intersecting the real axis at $t = -1$ and $t = +1$.

Similarly, the unit circle in the t-plane is projected stereographically onto the equatorial plane of the Riemann sphere.

Penrose comments that by rotating the sphere through 90° about the real axis of the t-plane, the circle on the Riemann sphere corresponding to the real axis of the t-plane and the circle on the Riemann sphere corresponding to the unit circle in the t-plane, are interchanged. He goes on to say that this can be considered to be a holomorphic map of the sphere to itself, or if we prefer, to another copy of the sphere. He then says that this particular rotation can be exhibited as a relation between the Riemann sphere of the t-plane and the Riemann sphere of the z-plane, by the bilinear transformation:

$$z = \frac{i-t}{i+t}, \; t = \frac{z-1}{iz+i}.$$

In other words, if we transform the t-plane into the z-plane by means of this transformation and then construct the Riemann spheres from the two extended complex planes for t and z, we will find that the relation between the two spheres is the rotation through 90° described above.
See figure 1 for a pictorial representation of this idea. He then asks us to show that this is indeed the case.

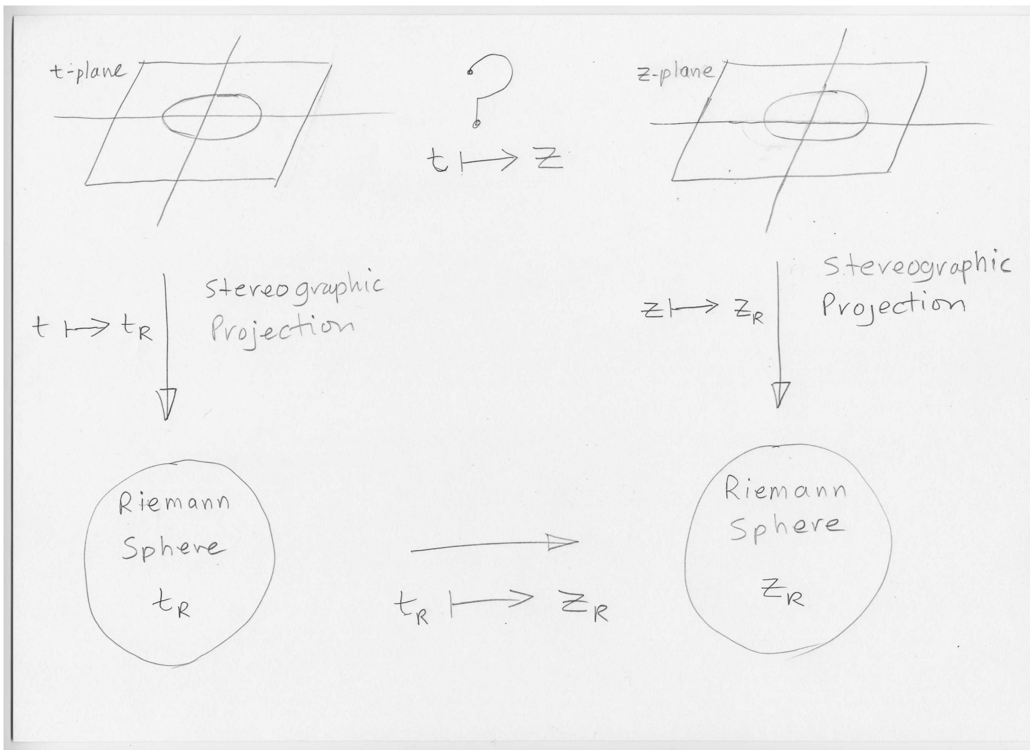

Figure 1: The transformations of the complex planes and Riemann spheres

The explanation
Figure 2 shows the t-plane and its associated Riemann sphere.
The right hand hemisphere in the figure contains all the points projected from the upper half of the t plane onto the Riemann sphere by stereographic projection - those from inside and on the unit circle are projected onto the lower half of this hemisphere, and those from outside the unit circle are projected onto the upper half of the hemisphere, and vice versa for the left hand hemisphere.

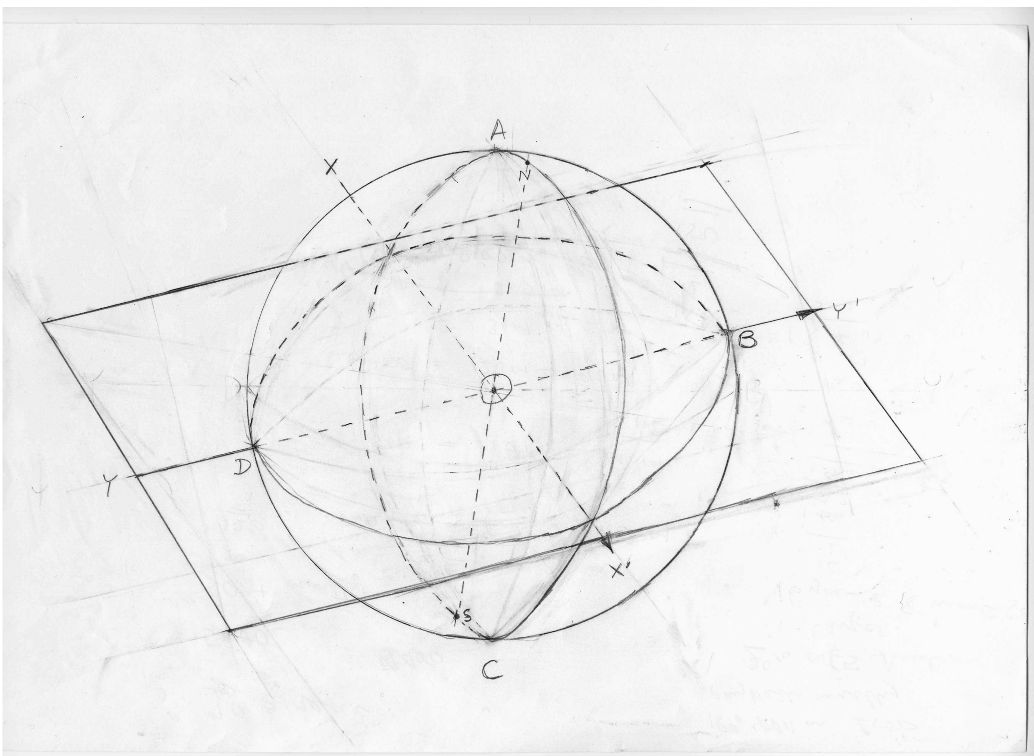

Figure 2: Complex plane and associated Riemann sphere

If we were to rotate the sphere clockwise, looking from X' towards the origin, through 90° about the real axis of the complex plane, then the right hand hemisphere would become the bottom hemisphere, and the real axis and unit circle change places.
The bottom hemisphere would now have all the points on and inside the

unit circle of the new complex plane z projected onto it by stereographic projection.

Furthermore, the left hand hemisphere would become the top hemisphere which would now have all the points outside the unit sphere of the new complex plane z projected stereographically upon it.

For these 2 things to happen we must:

1. **Transform the upper half of the original complex plane t into the inside of the unit circle on our new complex plane z.**

2. **Transform the lower half of the original complex plane t to the outside of the unit circle in the new complex plane z.**

By looking again at version c of the solution to exercise 8.4 in the RTR forum, we can remind ourselves that the transformation $z \mapsto 1/z$ transforms circles to circles, except for those circles which pass through the origin, which are transformed to straight lines.

In particlar, those circles whose centres lie on the axes and which pass through the origin are transformed to the co-ordinate gridlines.

Since $z \mapsto 1/z$ *is its own inverse*, it follows that the gridlines of the complex plane are transformed to circles centred on the real and imaginary axes of the complex plane and which pass through the origin.
This is illustrated in RTR figure 8.6.
Using the solution to 8.4 again, we can see that the transformation $z \mapsto 1/t$ would transform the strip above and including the line through $t = i$, parallel to the real axis, onto and inside the circle centred at $z = -i/2$ and which passes through the origin.
The strip between the real axis and the line through $t = i$ parallel to the real axis is mapped to that part of the lower half-plane outside this same circle.

Similarly the lower half-plane of the t-plane is transformed to the upper half-plane of the z-plane.
However this is not quite what is required to achieve the mappings 1 and 2 above.
We need the *whole* of the upper half of the t-plane to be transformed onto and inside the circle centred at $z = -i/2$, and the *lower* half-plane of the t-plane to be transformed to the rest of the complex plane. Then we can just transform the *whole* of this new plane so that circle becomes the unit circle centred at the origin of the z-plane.
We must therefore translate the whole plane one unit in the positive direction of the imaginary axis *before* we apply the transformation $1/z$, so that the *entire* upper half-plane of the t-plane is then transformed into the circle.
So the first transformation we apply is $t \mapsto t_1 = t + i$ followed by $t_1 \mapsto t_2 = 1/t_1$
If we now rotate the whole t_2-plane through 90° anticlockwise and then multiply by 2 we have the unit circle centred at $z = 1$. By translating everything one unit to the left we now have the unit circle centred on the origin. This is achieved by the transformation $t_2 \mapsto z = 2it_2 - 1$
We have now achieved what is required by applying the following 3 transformations in sequence:

$$t \mapsto t_1 = t + i,\ t_1 \mapsto t_2 = 1/t_1,\ t_2 \mapsto z = 2it_2 - 1$$

If we combine the first 2 transformations we obtain

$$t_2 = \frac{1}{t_1} = \frac{1}{t + i}$$

Then applying the final transformation we obtain

$$z = 2it_2 - 1 = \frac{2i}{t + i} - 1 = \frac{2i - t - i}{t + i} = \frac{i - t}{i + t}$$

which is what we wanted to show.

By substituting $t = \dfrac{z - 1}{iz + i}$ into $\dfrac{i - t}{i + t}$

we obtain z which confirms that this is the inverse transformation.